Systems Dynamics

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Lecture 2
State and Output Movement of Linear Discrete-Time Systems

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General State-Space Solution

General State-Space Solution

Consider a linear discrete-time free (no inputs) dynamic system:

$$x(k+1) = A(k)x(k)\,, \quad x(k_0) = x_0$$

Clearly, $x(k),\ k > k_0$ can be determined by **iterating** the state equation:

$$\begin{aligned}
&x(k_0) = x_0\\
&x(k_0+1) = A(k_0)x(k_0)\\
&x(k_0+2) = A(k_0+1)x(k_0+1) = A(k_0+1)A(k_0)x(k_0)\\
&\qquad\qquad\vdots\\
&x(k) = A(k-1)A(k-2)A(k-3)\cdots A(k_0+1)A(k_0)x(k_0)
\end{aligned}$$

Hence:

$$x(k) = \varphi(k, k_0, x_0) = \Phi(k, k_0)x_0$$

where the **discrete-time state-transition matrix** is:

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A(j)\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

General State-Space Solution (cont.)

Now, consider a linear discrete-time dynamic system with inputs:

$$x(k+1) = A(k)x(k) + B(k)u(k)\,, \quad x(k_0) = x_0$$

Clearly:

$$\begin{aligned}
&x(k_0) = x_0 \\
&x(k_0+1) = A(k_0)x(k_0) + B(k_0)u(k_0) \\
&x(k_0+2) = A(k_0+1)x(k_0+1) + B(k_0+1)u(k_0+1) \\
&\quad = A(k_0+1)[A(k_0)x(k_0) + B(k_0)u(k_0)] + B(k_0+1)u(k_0+1) \\
&\quad = A(k_0+1)A(k_0)x(k_0) + A(k_0+1)B(k_0)u(k_0) + B(k_0+1)u(k_0+1) \\
&x(k_0+3) = A(k_0+2)x(k_0+2) + B(k_0+2)u(k_0+2) \\
&\quad = A(k_0+2)A(k_0+1)A(k_0)x(k_0) + A(k_0+2)A(k_0+1)B(k_0)u(k_0) \\
&\quad +A(k_0+2)B(k_0+1)u(k_0+1) + B(k_0+2)u(k_0+2) \\
&\qquad\qquad \vdots
\end{aligned}$$

General State-Space Solution (cont.)

Therefore, using

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A(j)\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

one gets

$$\begin{aligned} x(k) &= \varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\}) \\ &= \Phi(k, k_0)x_0 + \sum_{j=k_0}^{k-1} \Phi(k, j+1)B(j)u(j)\,, \quad k > k_0 \end{aligned}$$

which expresses the **general solution** providing the state movement of a linear discrete-time dynamic system.

The determination of the state transition matrix $\Phi(k, k_0)$ is clearly very important.

General State-Space Solution (cont.)

- **Free state movement**. Setting $u(k) = 0, \forall\, k \geq k_0$ gives:

$$x(k) = \varphi(k, k_0, x_0, 0) = \varphi_L(k) = \Phi(k, k_0)x_0\,, \quad k > k_0$$

- **Forced state movement**. Setting $x_0 = 0$ gives:

$$\begin{aligned} x(k) \quad &= \varphi(k, k_0, 0, \{u(k_0), \ldots, u(k-1)\}) = \varphi_F(k) \\ &= \sum_{j=k_0}^{k-1} \Phi(k, j+1)B(j)u(j)\,, \quad k > k_0 \end{aligned}$$

The **total state movement** is thus given by:

$$\varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\}) = \varphi_L(k) + \varphi_F(k)$$

which is a direct consequence of the **linearity** of the dynamic system.

General State-Space Solution (cont.)

Now, let us add the output equation:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k)\,, \quad x(k_0) = x_0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

one gets:

$$\begin{aligned} y(k) = \quad & C(k)\Phi(k,k_0)x_0 \\ & + \sum_{j=k_0}^{k-1} C(k)\Phi(k,j+1)B(j)u(j) + D(k)u(k)\,, \quad k > k_0 \end{aligned}$$

- **Free output movement**. Setting $u(k) = 0, \forall\, k \geq k_0$ gives:

$$y(k) = y_L(k) = C(k)\Phi(k,k_0)x_0\,,\ k > k_0$$

- **Forced output movement**. Setting $x_0 = 0$ gives:

$$y(k) = y_F(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k,j+1)B(j)u(j) + D(k)u(k)\,,\ k > k_0$$

The **total output movement** is thus given by:

$$y(k) = y_L(k) + y_F(k)$$

State-Space Solution: the Time-Invariant Case

State-Space Solution: the Time-Invariant case

- In the time-invariant case, matrices $A(k), B(k), C(k), D(k)$ do not depend on time-index k, that is they are **constant** matrices A, B, C, D.
- Hence, when considering a linear discrete-time free (no inputs) time-invariant dynamic system:

$$x(k+1) = Ax(k)\,, \quad x(k_0) = x_0$$

one gets:

$$x(k) = \varphi(k, k_0, x_0) = \Phi(k, k_0)x_0$$

where the **discrete-time state-transition matrix** now takes on the form

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A = A^{(k-k_0)}\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

- With some abuse of notation, we denote $\Phi(k - k_0)$ to highlight the dependence on $(k - k_0)$ instead of k and k_0 separately.

State-Space Solution: the Time-Invariant case (cont.)

Now, consider a linear discrete-time time-invariant dynamic system with inputs:

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0$$

Therefore, using

$$\Phi(k-k_0) = A^{(k-k_0)}\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

one gets

$$\begin{aligned} x(k) \quad &= \varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\}) \\ &= A^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} A^{k-(j+1)}Bu(j)\,, \quad k > k_0 \end{aligned}$$

The explicit form $\Phi(k-k_0) = A^{(k-k_0)}$ will be used later on to determine the state and output evolution over time in **closed-form**.

State-Space Solution: the Time-Invariant case (cont.)

- **Free state movement**. Setting $u(k) = 0, \forall\, k \geq k_0$ gives:

$$x(k) = \varphi(k, k_0, x_0, 0) = \varphi_L(k) = A^{(k-k_0)} x_0\,, \quad k > k_0$$

- **Forced state movement**. Setting $x_0 = 0$ gives:

$$\begin{aligned} x(k) \quad &= \varphi(k, k_0, 0, \{u(k_0), \ldots, u(k-1)\}) = \varphi_F(k) \\ &= \sum_{j=k_0}^{k-1} A^{k-(j+1)} B u(j)\,, \quad k > k_0 \end{aligned}$$

The **total state movement** is thus given by:

$$\varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\}) = \varphi_L(k) + \varphi_F(k)$$

which is a direct consequence of the **linearity** of the dynamic system.

State-Space Solution: the Time-Invariant case (cont.)

Now, by adding the output equation:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

one gets:

$$\begin{aligned} y(k) = \quad & CA^{(k-k_0)}x_0 \\ & + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k)\,, \quad k > k_0 \end{aligned}$$

- **Free output movement**. Setting $u(k) = 0, \forall\, k \geq k_0$ gives:

$$y(k) = y_L(k) = CA^{(k-k_0)}x_0\,,\ k > k_0$$

- **Forced output movement**. Setting $x_0 = 0$ gives:

$$y(k) = y_F(k) = \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k)\,,\ k > k_0$$

The **total output movement** is thus given by:

$$y(k) = y_L(k) + y_F(k)$$

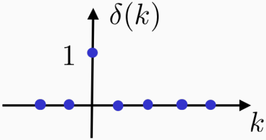
Input-Output Dynamic Description for Linear Systems

Input-Output Dynamic Description of Linear Systems

Preliminaries

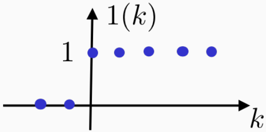
Discrete-time unit **impulse** sequence

$$\delta(k) = \begin{cases} 0, & k \neq 0,\ k \in \mathbb{Z} \\ 1, & k = 0 \end{cases}$$



Discrete-time unit **step** sequence

$$1(k) = \begin{cases} 0, & k < 0,\ k \in \mathbb{Z} \\ 1, & k \geq 0,\ k \in \mathbb{Z} \end{cases}$$



$$\Longrightarrow \quad \delta(k) = 1(k) - 1(k-1)\,; \quad 1(k) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \delta(k-j), & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Moreover, an arbitrary sequence $\{x(k)\}$ can be expressed as

$$x(k) = \sum_{j=-\infty}^{\infty} x(j)\delta(k-j)$$

Input-Output Dynamic Description of Linear Systems (cont.)

- Consider a linear discrete-time system with scalar input and output

- Moreover, consider the ”external” input/output relationship

$$y(k) = \sum_{j=-\infty}^{\infty} h(k,j)u(j) \quad (\star)$$

Assumption. The sequences $\{h(k,j)\}$ for any given k and $\{u(j)\}$ are such that the relationship $(\star)$ is well-defined. For example, $\{h(k,j)\} \in l_2$ and $\{u(j)\} \in l_2$.

- Under the above assumption, relationship $(\star)$ is **linear**.

Input-Output Dynamic Description of Linear Systems (cont.)

- Denote by $h(k, j)$ the output response at time k produced by a unit impulse $\delta(j)$ applied at time j
- By linearity, the output response at time k produced by a impulse of amplitude $u(j)$ applied at time j is $h(k, j)u(j)$
- By linearity, the output response at time k produced by two impulses of amplitude $u(j_1)$ and $u(j_2)$ applied at times j_1 and j_2, respectively, is $h(k, j_1)u(j_1) + h(k, j_2)u(j_2)$

Input-Output Model

At time k, the system output $y(k)$ produced by the input sequence $\{u(j)\}$ is given by

$$y(k) = \sum_{j=-\infty}^{\infty} h(k, j)u(j)$$

where $h(k, j)$ denotes the output response at time k produced by a unit impulse $\delta(k - j)$ applied at time j

Input-Output Dynamic Description of Linear Systems (cont.)

Properties

- Due to **causality**, the response to an input sequence has to be **identically zero before the input sequence is applied**. Hence:

$$h(k,j) = 0\,, \quad \forall\, j, \forall\, k < j$$

Hence:

$$y(k) = \sum_{j=-\infty}^{k} h(k,j)u(j)$$

$$\begin{aligned} \Longrightarrow \quad y(k) &= \sum_{j=-\infty}^{k_0-1} h(k,j)u(j) + \sum_{j=k_0}^{k} h(k,j)u(j) \\ &= Y(k;k_0-1) + \sum_{j=k_0}^{k} h(k,j)u(j) \end{aligned}$$

Input-Output Dynamic Description of Linear Systems (cont.)

- The system is **at rest** at time k_0 if

$$u(k) = 0, \forall\, k \geq k_0 \quad \Longrightarrow \quad y(k) = 0, \forall\, k \geq k_0$$

  and this implies $Y(k; k_0 - 1) = 0$.
- Hence, if the system is **at rest** at time k_0, it follows that

$$y(k) = \sum_{j=k_0}^{\infty} h(k,j)u(j)$$

  and due to causality, one gets

$$y(k) = \sum_{j=k_0}^{k} h(k,j)u(j)$$

Input-Output Dynamic Description of Linear Systems (cont.)

- If the system is **time-invariant**, denoting by $\{h(k,0)\}$ the response to $\{\delta(k)\}$, it follows that $\{h(k-j,0)\}$ is the response to $\{\delta(k-j)\}$
- Letting (with some abuse of notation)

$$h(k-j) := h(k-j,0)$$

one gets the well-known **convolution formula**:

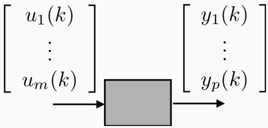
$$y(k) = u(k) * h(k) = \sum_{j=-\infty}^{\infty} h(k-j)u(j)$$

or equivalently (via a change of variables)

$$y(k) = h(k) * u(k) = \sum_{i=-\infty}^{\infty} h(i)u(k-i)$$

Input-Output Dynamic Description of Linear Systems (cont.)

- Consider a linear discrete-time system with **vector input and output**



- The scalar case (with all properties) can be generalised as:

$$y(k) = \sum_{j=-\infty}^{\infty} H(k,j)u(j)$$

$$H(k,j) = \begin{bmatrix} h_{11}(k,j) & h_{12}(k,j) & \cdots & h_{1m}(k,j) \\ h_{21}(k,j) & h_{22}(k,j) & \cdots & h_{2m}(k,j) \\ \cdots & \cdots & \cdots & \\ h_{p1}(k,j) & h_{p2}(k,j) & \cdots & h_{pm}(k,j) \end{bmatrix}$$

where $h_{rs}(k,j)$ denotes the r-th component of the response at time k produced by a unit impulse applied at time j on the s-th component of the input, while all other input components are set to zero.

Relationship between State-Space and Input-Output Dynamic Descriptions

Consider a state-space description with **initial state set to zero**:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k)\,, \quad x(k_0) = 0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

Recalling that

$$y(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k)\,, \quad k > k_0$$

one gets immediately

$$H(k, j) = \begin{cases} C(k)\Phi(k, j+1)B(j)\,, & k > j \\ D(k) & k = j \\ 0 & k < j \end{cases}$$

which, in the **time-invariant** case, becomes

$$H(k-j) = \begin{cases} CA^{k-(j+1)}B\,, & k > j \\ D & k = j \\ 0 & k < j \end{cases}$$

Determination of the State/Output Movement

Determination of the State/Output Movement

Response Modes

Determination of the State/Output Movement

Recall that in the general **time-varying** case one has:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k)\,, \quad x(k_0) = x_0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

one gets:

$$\begin{aligned} y(k) = \quad & C(k)\Phi(k,k_0)x_0 \\ & + \sum_{j=k_0}^{k-1} C(k)\Phi(k,j+1)B(j)u(j) + D(k)u(k)\,, \quad k > k_0 \end{aligned}$$

where

$$\Phi(k,k_0) = \prod_{j=k_0}^{k-1} A(j)\,, \quad k > k_0\,; \quad \Phi(k_0,k_0) = I$$

is the **state-transition matrix**.

Determination of the State/Output Movement (cont.)

In the **time-invariant** case, recall that the solution specialises as follows:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

one gets:

$$\begin{aligned} y(k) = \quad & CA^{(k-k_0)}x_0 \\ & + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k)\,, \quad k > k_0 \end{aligned}$$

where the **state-transition matrix** now is given by:

$$\Phi(k-k_0) = A^{(k-k_0)}\,, \quad k > k_0\,; \quad \Phi(k_0, k_0) = I$$

Response Modes

- Without loss of generality we let $k_0 = 0$ and we ”expand” matrix $A^{k-k_0} = A^k$ in ”matrix partial fractions”.
- Clearly
$$\det(zI - A) = \prod_{i=1}^{\sigma} (z - \lambda_i)^{n_i}$$
where $\lambda_1, \ldots, \lambda_\sigma$ are the **distinct** eigenvalues of A and n_i is the **algebraic multiplicity** of such eigenvalues.
- Of course $\displaystyle\sum_{i=1}^{\sigma} n_i = n$.
- It can be shown that:
$$A^k = \sum_{i=1}^{\sigma} \left[A_{i0}\lambda_i^k 1(k) + \sum_{l=0}^{n_i-1} A_{il} k(k-1)\cdots(k-l+1)\lambda_i^{k-l} 1(k-l) \right]$$
where
$$A_{il} = \frac{1}{l!}\frac{1}{(n_i-1-l)!}\lim_{z\to\lambda_i}\left\{ \frac{d^{n_i-1-l}}{dz^{n_i-1-l}} \left[(z-\lambda_i)^{n_i}(zI-A)^{-1} \right] \right\}$$

Response Modes (cont.)

Hence:

- A^k can be expressed as a sum of terms $A_{il}k!\begin{pmatrix} k \\ l \end{pmatrix}\lambda_i^k$ which are called **Response Modes**
- If an eigenvalue λ_i has algebraic multiplicity n_i, then, in general, n_i response modes

$$A_{il}k!\begin{pmatrix} k \\ l \end{pmatrix}\lambda_i^k,\, l = 0, 1, \ldots, n_i - 1$$

  can be associated to λ_i.
- When all eigenvalues of A are distinct, one has $\sigma = n;\, n_i = 1, i = 1, \ldots, n$ and

$$A^k = \sum_{i=1}^{n} A_i \lambda_i^k$$

with

$$A_i = \lim_{z \to \lambda_i} \left[(z - \lambda_i)(zI - A)^{-1}\right]$$

Response Modes: A different Characterisation

In the special case of **distinct eigenvalues** of A:

- In such a case: $\det(zI - A) = \prod_{i=1}^{n}(z - \lambda_i)$ and $A^k = \sum_{i=1}^{n} A_i \lambda_i^k$
- It can be shown that $A_i = v_i \tilde{v}_i^\top$ where:
  - $(\lambda_i I - A)v_i = 0$: v_i right eigenvector associated with λ_i
  - $\tilde{v}_i^\top(\lambda_i I - A) = 0$: $\tilde{v}_i^\top$ left eigenvector associated with λ_i

  In fact:

$$Q := [v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n] \implies P = Q^{-1} = \begin{bmatrix} \tilde{v}_1^\top \\ \vdots \\ \tilde{v}_n^\top \end{bmatrix}; \; \tilde{v}_i^\top v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

  and then

$$\begin{aligned} (zI - A)^{-1} &= [zI - Q\mathrm{diag}\,[\lambda_1, \ldots, \lambda_n]Q^{-1}]^{-1} \\ &= Q[zI - \mathrm{diag}\,[\lambda_1, \ldots, \lambda_n]]^{-1}Q^{-1} \\ &= Q\mathrm{diag}\,[(z - \lambda_1)^{-1}, \ldots, (z - \lambda_n)^{-1}]Q^{-1} = \sum_{i=1}^{n} v_i \tilde{v}_i^\top (z - \lambda_i)^{-1} \end{aligned}$$

Response Modes: A different Characterisation (cont.)

- If the initial state vector x_0 is "parallel" to eigenvector v_j of A, then the only response mode showing up int the state movement is λ_j^k:

$$x_0 = \alpha v_j \implies x(k) = A^k x_0 = v_1 \tilde{v}_1^\top x_0 \lambda_1^k + \cdots + v_n \tilde{v}_n^\top x_0 \lambda_n^k = \alpha v_j \lambda_j^k$$

Example: consider $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$; $\lambda_1 = -1,\ \lambda_2 = 1$

$$\implies Q = [v_1 \mid v_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ Q^{-1} = \begin{bmatrix} \tilde{v}_1^\top \\ \tilde{v}_2^\top \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^k = v_1 \tilde{v}_1^\top \lambda_1^k + v_2 \tilde{v}_2^\top \lambda_2^k = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} (-1)^k + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} 1^k$$

and thus, if $x_0 = \alpha v_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then the response mode 1^k **does not show up** in the free state response starting from such an initial state x_0

Calculation of A^k by Similarity Transformation

Consider:

- $x(k+1) = Ax(k)$, $x(0) = x_0 \implies x(k) = A^k x_0$
- $T \in \mathbb{R}^{n\times n},\ \det(T) \neq 0 \implies x = T\hat{x},\quad \hat{x} = T^{-1}x$

Hence $\hat{x}(k+1) = T^{-1}Ax(k) = T^{-1}AT\hat{x}(k)$, $\hat{x}_0 = T^{-1}x_0$ which yields

$$\hat{x}(k) = \left(T^{-1}AT\right)^k T^{-1}x_0$$

Letting $J := T^{-1}AT$, one gets the closed-form expression for the free-state response expressed in the original state coordinates

$$x(k) = TJ^kT^{-1}x_0$$

Suppose now that the similarity transformation is such that

$$J = T^{-1}AT$$

takes on the **Jordan Canonical Form**.

Calculation of A^k by Similarity Transformation (cont.)

Case 1. Suppose that matrix A admits the construction of a basis of n linearly-independent eigenvectors v_i associated with the eigenvalues λ_i, $i = 1, \ldots, n$ (**not necessarily distinct**).

Thus:

$$T = [v_1 | v_2 | \cdots | v_n] \implies J = T^{-1}AT = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Hence:

$$J^k = \begin{bmatrix} {\lambda_1}^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & {\lambda_n}^k \end{bmatrix}$$

$$\implies x(k) = TJ^kT^{-1}x_0 = T \begin{bmatrix} {\lambda_1}^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & {\lambda_n}^k \end{bmatrix} T^{-1}x_0$$

Calculation of A^k by Similarity Transformation (cont.)

Case 2. Consider the general case in which matrix A has multiple eigenvalues. It is always possible to construct a basis of n linearly-independent vectors v_i such that:

$$T = [v_1|v_2|\cdots|v_n] \implies J = T^{-1}AT = \begin{bmatrix} J_0 & \cdots & \cdots & 0 \\ \vdots & J_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & J_s \end{bmatrix}$$

where

$$J_0 = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{bmatrix}$$

Calculation of A^k by Similarity Transformation (cont.)

and $J_i,\, i \geq 1$ is a $n_i \times n_i$ matrix taking on the special form

$$J_i = \begin{bmatrix} \lambda_{k+i} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{k+i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda_{k+i} \end{bmatrix}$$

where **not necessarily** $\lambda_{k+i} \neq \lambda_{k+j},\, i \neq j$ and

$$k + n_1 + \cdots + n_s = n$$

Matrix J is block-diagonal and its special structure makes it possible to compute A^k in **closed-form**.

Calculation of A^k by Similarity Transformation (cont.)

In fact:

$$J^k = \begin{bmatrix} J_0^k & \cdots & \cdots & 0 \\ & {J_1}^k & & \\ & & \ddots & \\ 0 & \cdots & \cdots & {J_s}^k \end{bmatrix}$$

where

$$ {J_0}^k = \begin{bmatrix} {\lambda_1}^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & {\lambda_r}^k \end{bmatrix}$$

Then:

$$x(k) = TJ^kT^{-1}x_0 = T \begin{bmatrix} J_0^k & \cdots & \cdots & 0 \\ & {J_1}^k & & \\ & & \ddots & \\ 0 & \cdots & \cdots & {J_s}^k \end{bmatrix} T^{-1}x_0$$

Calculation of A^k by Similarity Transformation (cont.)

Concerning the computation of $J_i^k,\, i = 1, \ldots, s$ we can write:

$$J_i = \lambda_{r+i} I_i + N_i$$

where I_i is the identity matrix with dimension $n_i \times n_i$ and N_i is a matrix of dimension $n_i \times n_i$ having the form:

$$N_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Matrix N_i is a nilpotent matrix, that is, it holds:

$$N_i^k = 0,\, \forall k \geq n_i$$

Calculation of A^k by Similarity Transformation (cont.)

On the other hand, one immediately gets:

$$\begin{aligned} J_i^k &= (\lambda_{r+i} I_i + N_i)^k \\ &= \lambda_{r+i}^k I + k\lambda_{r+i}^{k-1} N_i + \frac{k(k-1)}{2!}\lambda_{r+i}^{k-2} N_i^2 + \cdots + k\lambda_{r+i} N_i^{k-1} + N_i^k \end{aligned}$$

thus getting to **discrete-time response modes** of the form

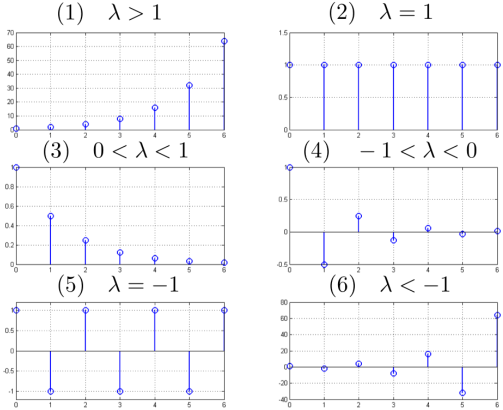
$$\lambda^k, \begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$$

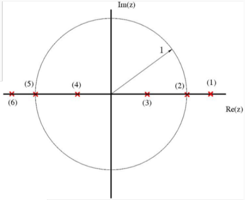
Determination of the State/Output Movement

Qualitative Behaviour of Response Modes

Qualitative Behaviour of Response Modes

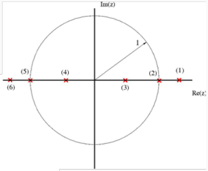
- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{R}$, multiplicity $= 1$

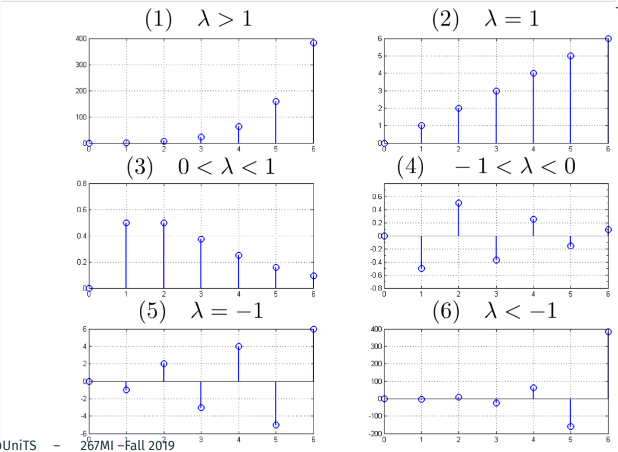




Qualitative Behaviour of Response Modes

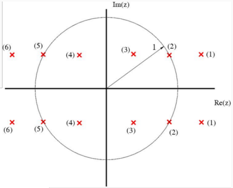
- $\binom{k}{n_i} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{R}$, multiplicity > 1

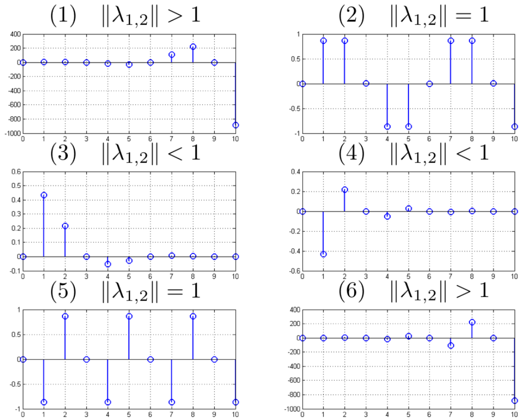




Qualitative Behaviour of Response Modes

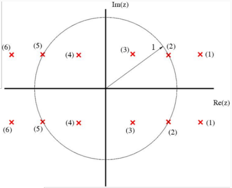
- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{C}$, multiplicity $= 1$

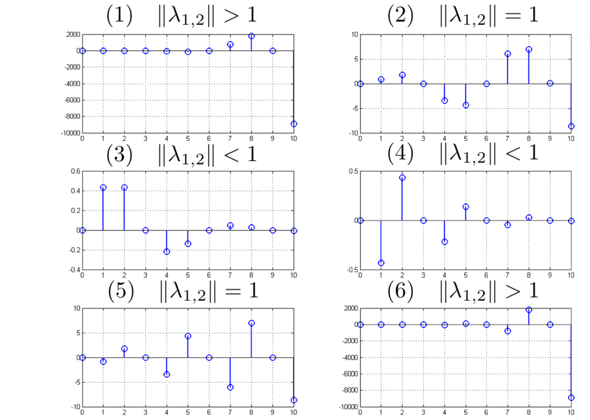




Qualitative Behaviour of Response Modes

- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{C}$, multiplicity > 1





External Description of LTI Dynamic Systems: Transfer Function

External Description of LTI Dynamic Systems: Transfer Function

Recall the relationship between the state space description and the impulse response **(an external description)**:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k)\,, & x(k_0) = 0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

Recalling that

$$y(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k)\,, \quad k > k_0$$

one gets immediately

$$H(k, j) = \begin{cases} C(k)\Phi(k, j+1)B(j)\,, & k > j \\ D(k) & k = j \\ 0 & k < j \end{cases}$$

which, in the **time-invariant** case, becomes

$$H(k-j) = \begin{cases} CA^{k-(j+1)}B\,, & k > j \\ D & k = j \\ 0 & k < j \end{cases}$$

Transfer Function

Consider the time-invariant dynamic system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k)\,, \quad x(k_0) = 0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Applying the $\mathcal{Z}$ Transform to both sides one gets:

$$\begin{aligned} & z\left[X(z) - x_0\right] = AX(z) + BU(z) \\ & \quad \Longrightarrow (zI - A)X(z) = z\,x_0 + BU(z) \\ & \quad \Longrightarrow \begin{cases} X(z) = (zI-A)^{-1} z\,x_0 + (zI-A)^{-1}BU(z) \\ \\ Y(z) = CX(z) + DU(z) \end{cases} \\ & \quad \Longrightarrow Y(z) = C(zI-A)^{-1} z\,x(0) + \left[C(zI-A)^{-1}B + D\right] U(z) \end{aligned}$$

Letting $x_0 = 0$, it follows that:

$$Y(z) = \left[C(zI-A)^{-1}B + D\right]U(z) = H(z)U(z)$$

and $H(z)$ is called **transfer function**.

Transfer Function (cont.)

Let's analyse the structure of the transfer function:

$$H(z) = \begin{bmatrix} H_{11}(z) & \cdots & H_{1m}(z) \\ \vdots & & \vdots \\ H_{i1}(z) & \cdots & H_{im}(z) \\ \vdots & & \vdots \\ H_{p1}(z) & \cdots & H_{pm}(z) \end{bmatrix}$$

$H(z)$ is a $p \times m$ matrix where the i-th component of the output vector is given by:

$$Y_i(z) = \sum_{j=1}^{m} H_{ij}(z)U_j(z) = H_{i1}(z)U_1(z) + H_{i2}(z)U_2(z) + \cdots$$

Hence:

$$\begin{aligned} x(0) &= x_0 \\ u_r(k) &= 0, \;\; r \neq j \end{aligned} \qquad \Longrightarrow \qquad H_{ij}(z) = \frac{Y_i(z)}{U_j(z)}$$

Transfer Function of equivalent dynamic systems

Recall:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Let $\hat{x} := T^{-1}x$, where $T \in \mathbb{R}^{n \times n}$ is a generic non-singular $n \times n$ matrix ($\det(T) \neq 0$). Then, the equivalent state-space description is given by:

$$\begin{cases} \hat{x}(k+1) = T^{-1}x(k+1) = T^{-1}AT\hat{x}(k) + T^{-1}Bu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = CT\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Hence:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \iff \begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Transfer Function of equivalent dynamic systems (cont.)

$$
\begin{aligned}
\hat{H}(z) &= \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \\
&= C\left[T\left(zI - T^{-1}AT\right)^{-1}T^{-1}\right]B + D \\
&= C\left[T\left(zT^{-1}T - T^{-1}AT\right)^{-1}T^{-1}\right]B + D \\
&= C\left[T\left(T^{-1}(zI - A)T\right)^{-1}T^{-1}\right]B + D \\
&= C\left[TT^{-1}\left(zI - A\right)^{-1}TT^{-1}\right]B + D \\
&= C\left[\left(zI - A\right)^{-1}\right]B + D \\
&= H(z)
\end{aligned}
$$

Hence: the transfer function does not depend on the specific choice of the state variables

Transfer Function: Properties

Consider the scalar case, that is, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$:

$$H(z) = C\left[(zI - A)^{-1}\right]B + D$$

and

$$(zI - A)^{-1} = \begin{bmatrix} z - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & z - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & z - a_{nn} \end{bmatrix}^{-1}$$

Transfer Function: Properties (cont.)

The inverse can be expressed as:

$$(zI - A)^{-1} = \frac{1}{\det(zI - A)} K(z)$$

where $K(z)$ is the matrix of the algebraic complements.

Clearly:

- $\det(zI - A)$ is a polynomial with degree n
- $K(z) = [k_{ij}(z);\, i, j = 1, \ldots, n]$ where $k_{ij}(z)$ is a polynomial of degree $< n, \forall\, i, j$
- $C(zI - A)^{-1} B = \dfrac{1}{\det(zI - A)} CK(z)B = \dfrac{M(z)}{\varphi(z)}$ where $M(z)$ is a polynomial of degree $< n,$

Transfer Function: Properties (cont.)

Therefore:

$$
\begin{aligned}
H(z) = C\left(zI - A\right)^{-1} B + D &= \frac{M(z)}{\varphi(z)} + D \\
&= \frac{M(z) + D\varphi(z)}{\varphi(z)} = \frac{N(z)}{\varphi(z)}
\end{aligned}
$$

where:

- $N(z)$ in general is a polynomial of degree n
- In case of a **strictly proper** system, that is $D = 0$, $N(z)$ in general is a polynomial of degree $< n$
- All the above holds if **no common factors** between $N(z)$ and $\varphi(z)$ are present

Transfer Function: Properties (cont.)

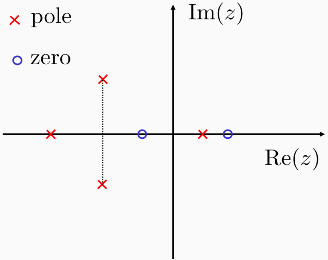
In the presence of common factors between $N(z)$ and $\varphi(z)$:

$$H(z) = \frac{\overline{N}(z)}{\overline{\varphi}(z)}$$

- $\overline{\varphi}(z)$ is a factor of $\varphi(z)$ of degree $\nu < n$
- $\overline{N}(z)$ has degree $m < \nu$ and has degree ν only if $D \neq 0$ (non strictly proper systems)

Transfer Function: Poles and Zeros (scalar case)

- **Poles**: roots of polynomial $\varphi(z)$
- **Zeros**: roots of polynomial $N(z)$



- The poles are eigenvalues of A
- An eigenvalue of A might not belong to the set of poles when common factors are present
- In case of more then one input and/or more than one output extra-care has to be exercised

Transfer Function: Example

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\[2em] y(k) = [0 \;\; 1]\, x(k) \end{cases} \qquad n = 2$$

Hence:

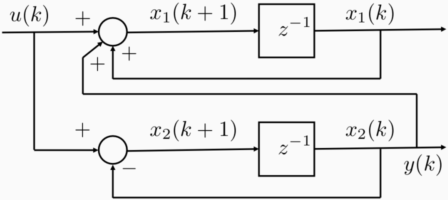
$$\begin{aligned} G(z) &= [0 \;\; 1] \begin{bmatrix} z-1 & -1 \\ 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [0 \;\; 1] \frac{1}{(z-1)(z+1)} \begin{bmatrix} z+1 & 1 \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{(z-1)}{(z-1)(z+1)} = \frac{1}{z+1} \end{aligned}$$

Thus: $\overline{\varphi}(z) = z + 1$ is a factor of $\varphi(z) = (z-1)(z+1)$

Transfer Function: Example (cont.)

The state equations have the form:

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) + u(k) \\ x_2(k+1) = -x_2(k) + u(k) \\ y(k) = x_2(k) \end{cases}$$



Only the dynamics $\begin{cases} x_2(k+1) = -x_2(k) + u(k) \\ y(k) = x_2(k) \end{cases}$ shows up in the transfer function $G(z) = \dfrac{1}{z+1}$ and the time-evolution of $x_1(k)$ is not influencing the output $y(k)$.

Transfer Function: Example in the Non-Scalar Case

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} u(k) \\ y(k) = [-3 \ \ 3]\, x(k) \end{cases}$$

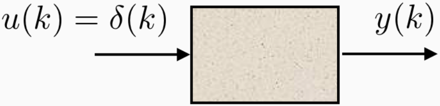
Hence, one gets:

$$\begin{aligned} H(z) &= [-3 \ \ 3] \begin{bmatrix} z & -1 \\ 1 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= [-3 \ \ 3]\, \frac{1}{(z+1)^2} \begin{bmatrix} z+2 & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{z+1} & \frac{3(z-1)}{(z+1)^2} \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{3(z-1)}{(z+1)^2} & \frac{3}{z+1} \end{bmatrix} \end{aligned}$$

The notion of zeros and poles of a transfer function in the non-scalar case is more complicated (and less useful though)

Transfer Function: Alternative Definition in the Scalar Case

$$
\begin{aligned}
&x(0) = 0 \\
&u(k) = \delta(k) \\
&\quad \Longrightarrow U(z) = \mathcal{Z}[\delta(k)] = 1
\end{aligned}
$$



Therefore:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{Y(z)}{1} = Y(z)$$

that is:

$$H(z) = \mathcal{Z}[\,\text{Impulse Response}\,]$$

Determination of Response Modes: Examples

Determination of Response Modes: Example 1

Consider:

$$\begin{cases} x(k+1) = \begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 2 & -1.5 \end{bmatrix} x(k) \end{cases}$$

Determine the free-state movement $x_l(k) = A^k\, x(0)$ starting from the initial state $x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$

The free-state movement is given by

$$x(k) = A^k\, x(0) + \sum_{i=0}^{k-1} A^{k-i-1}\, Bu(i)$$

We are going to determine the free-state movement in two ways:

- by the $\mathcal{Z}$ transform
- by calculating the matrix A^k.

Determination of Response Modes: Example 1 (cont.)

Calculation by the $\mathcal{Z}$ transform

$$x_l(k) = A^k\, x(0) \implies X_l(z) \;=\; z\,(z\,I \,-\, A)^{-1}\; x(0)$$

$$(z\,I \,-\, A) = \begin{bmatrix} z + 0.5 & -2 \\ 0 & z - 0.1 \end{bmatrix}$$

$$\implies (z\,I \,-\, A)^{-1} = \begin{bmatrix} \dfrac{2}{2\,z \,+\, 1} & \dfrac{40}{(2\,z \,+\, 1)\,(10\,z \,-\, 1)} \\ 0 & \dfrac{10}{10\,z \,-\, 1} \end{bmatrix}$$

Hence:

$$X_l(z) \;=\; \begin{bmatrix} \dfrac{20\,z\,(10\,z \,-\, 21)}{(10\,z \,-\, 1)\,(2\,z \,+\, 1)} \\ -\dfrac{100\,z}{10\,z \,-\, 1} \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

First, we proceed with the inverse $\mathcal{Z}$ transform:

$$X_l(z) = \begin{bmatrix} X_{l\,1}(z) \\ X_{l\,2}(z) \end{bmatrix} = \begin{bmatrix} \dfrac{20\,z\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)} \\ -\dfrac{100\,z}{10\,z - 1} \end{bmatrix}$$

Hence:

$$X_{l\,1}(z) = \frac{20\,z\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)}$$
$$\implies \frac{X_{l\,1}(z)}{z} = \frac{20\,(10\,z - 21)}{(10\,z - 1)\,(2\,z + 1)} = \frac{C_1}{z - \frac{1}{10}} + \frac{C_2}{z + \frac{1}{2}}$$

$$C_1 = \lim_{z \to \frac{1}{10}} \frac{20\,(10\,z - 21)}{10\,(2\,z + 1)} = -\frac{100}{3}\,;\; C_2 = \lim_{z \to -\frac{1}{2}} \frac{20\,(10\,z - 21)}{2\,(10\,z - 1)} = \frac{130}{3}$$

thus getting: $X_{l\,1}(z) = -\dfrac{100}{3}\,\dfrac{z}{\left(z - \frac{1}{10}\right)} + \dfrac{130}{3}\,\dfrac{z}{\left(z + \frac{1}{2}\right)}$

Determination of Response Modes: Example 1 (cont.)

Then, it follows that:

$$X_l(z) = \begin{bmatrix} -\dfrac{100}{3}\dfrac{z}{\left(z - \frac{1}{10}\right)} + \dfrac{130}{3}\dfrac{z}{\left(z + \frac{1}{2}\right)} \\ -10\dfrac{z}{\left(z - \frac{1}{10}\right)} \end{bmatrix}$$

and thus:

$$x_l(k) = \begin{bmatrix} \left\{-\dfrac{100}{3}\left(\dfrac{1}{10}\right)^k + \dfrac{130}{3}\left(-\dfrac{1}{2}\right)^k\right\} \cdot 1(k) \\ -10\left(\dfrac{1}{10}\right)^k \cdot 1(k) \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

Now, as alternative technique, we proceed with calculating the matrix A^k.

- $A = \begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix}$
- Eigenvalues: $\lambda_1 = -0.5$, $\lambda_2 = 0.1$. Hence, matrix A admits a diagonal similar matrix because the eigenvalues are distinct
- The characteristic polynomial is given by:

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda + 0.5)(\lambda - 0.1)$$

- A basis of linearly independent eigenvectors is now determined.

Determination of Response Modes: Example 1 (cont.)

- $Az = \lambda_1 z \quad \text{with} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$\begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -0.5 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \left\{ \begin{array}{rcl} -0.5z_1 + 2z_2 & = & -0.5z_1 \\ 0.1z_2 & = & -0.5z_2 \end{array} \right.$$

For example: $z_2 = 0 \implies z^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $Az = \lambda_2 z$

$$\begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.1 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \left\{ \begin{array}{rcl} -0.5z_1 + 2z_2 & = & 0.1z_1 \\ 0.1z_2 & = & 0.1z_2 \end{array} \right.$$

For example: $z_2 = \frac{3}{10} z_1 \implies z^{(2)} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

Determination of Response Modes: Example 1 (cont.)

One now proceeds with calculating the equivalent state-space representation of matrix A:

$$T = \left[z^{(1)} \middle| z^{(2)} \right] = \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} \implies T^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix}$$

thus obtaining:

$$\tilde{A} = T^{-1} A T = \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 2 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

The calculation of A^k is now straightforward:

$$A^k = M\tilde{A}^k M^{-1} = M \begin{bmatrix} \left(-\frac{1}{2}\right)^k & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} M^{-1}$$

$$= \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \left(-\frac{1}{2}\right)^k & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{1}{2}\right)^k & \left(-\frac{10}{3}\left(-\frac{1}{2}\right)^k + \frac{10}{3}\left(\frac{1}{10}\right)^k\right) \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix}$$

Determination of Response Modes: Example 1 (cont.)

Finally, from

$$A^k = \begin{bmatrix} \left(-\frac{1}{2}\right)^k & \left(-\frac{10}{3}\left(-\frac{1}{2}\right)^k + \frac{10}{3}\left(\frac{1}{10}\right)^k\right) \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix}$$

and $x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$, one gets:

$$x_l(k) = \begin{bmatrix} \left\{-\frac{100}{3}\left(\frac{1}{10}\right)^k + \frac{130}{3}\left(-\frac{1}{2}\right)^k\right\} \cdot 1(k) \\ -10\left(\frac{1}{10}\right)^k \cdot 1(k) \end{bmatrix}$$

Determination of Response Modes: Example 2

Consider:

$$\begin{cases} x_1(k+1) & = & x_1(k) + 4\,x_2(k) \\ x_2(k+1) & = & x_1(k) + x_2(k) \end{cases}$$

Setting $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, show in two different ways that

$$\lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = 2$$

We are going to determine the free-state movement yielding $x_1(k), x_2(k),\ \forall k \geq 0$ in two ways:

- by the $\mathcal{Z}$ transform
- by calculating the matrix A^k.

Determination of Response Modes: Example 2 (cont.)

Using the $\mathcal{Z}$ transform:

$$\begin{cases} zX_1(z) - z &= X_1(z) + 4X_2(z) \\ zX_2(z) - z &= X_1(z) + X_2(z) \end{cases} \implies \begin{cases} X_1(z) &= \dfrac{z(z+3)}{(z+1)(z-3)} \\ \\ X_2(z) &= \dfrac{z^2}{(z+1)(z-3)} \end{cases}$$

Hence:

$$\begin{cases} x_1(k) = \left[\left(-\dfrac{1}{2}\right)\,(-1)^k + \dfrac{3}{2}\;3^k\right] 1(k) \\ \\ x_2(k) = \left[\dfrac{1}{4}\,(-1)^k + \dfrac{3}{4}\;3^k\right] 1(k) \end{cases}$$

$$\implies \lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = \lim_{k\to\infty} \frac{\left(\dfrac{3}{2}\right)\;3^k}{\left(\dfrac{3}{4}\right)\;3^k} = 2$$

Determination of Response Modes: Example 2 (cont.)

Using the calculation of A^k:

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \Longrightarrow \det(\lambda I - A) = \lambda^2 - 2\lambda - 3 = 0 \Longrightarrow$$

distinct eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$

$$\begin{aligned} \ker(A - 3I) &= \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \\ \ker(A + I) &= \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \end{aligned} \Longrightarrow \begin{aligned} T &= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \\ T^{-1} &= -\frac{1}{4} \begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} \end{aligned}$$

Thus

$$\tilde{A} = T^{-1} A T = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

Determination of Response Modes: Example 2 (cont.)

By some algebra:

$$A^k = T\,\tilde{A}^k\,T^{-1} = \begin{bmatrix} \frac{1}{2}\,3^k + \frac{1}{2}\,(-1)^k & 3^k - (-1)^k \\ \frac{1}{4}\,\left(3^k - (-1)^k\right) & \frac{1}{2}\,3^k + \frac{1}{2}\,(-1)^k \end{bmatrix}$$

and then:

$$x(k) = A^k x(0) = \begin{cases} x_1(k) = \left[\left(-\frac{1}{2}\right)(-1)^k + \frac{3}{2}\,3^k\right] 1(k) \\ x_2(k) = \left[\frac{1}{4}\,(-1)^k + \frac{3}{4}\,3^k\right] 1(k) \end{cases}$$

$$\implies \lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = \lim_{k\to\infty} \frac{\left(\frac{3}{2}\right)\,3^k}{\left(\frac{3}{4}\right)\,3^k} = 2$$

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Lecture 2

State and Output Movement of Linear Discrete-Time Systems

END