Systems Dynamics

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Lecture 5

A (very) short glimpse on probability theory, random variables and discrete-time stochastic processes

Random variables

- **Random experiment**: analysis of characteristic elements of phenomena yielding unpredictable results.
- **Results space**: we denote by S the set of all possible results of the experiment. Result: $s \in S$.
- **Events**: sets of results of specific interest. Hence an event is a subset of S .

Random variable

Given a random experiment, a **random variable** (r.v.) is a variable $v(s)$ taking values depending on the result $s \in S$ of a random experiment via a function $\varphi(\cdot)$.

Probability distribution & density functions

Probability distribution function

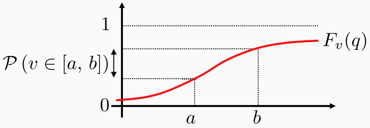
Provides information on the random variable v and it is defined as

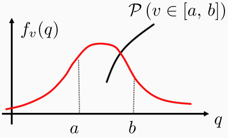
$$F_v(q) = \mathcal{P}\left(v \leq q\right)$$

According to the definition $\mathcal{P}\left(v \in [a,\, b]\right) = F_v(b) - F_v(a)$

Probability density function

$$f_v(q) = \frac{d\,F_v}{d\,q}$$





Clearly $\mathcal{P}\left(v \in [a,\, b]\right)$ is the area “under” the diagram of $f(q)$ in the interval $[a,\, b]$.

Functions of random variables

- **Expected value (average value, average)**

$$\mathrm{E}\left(v\right) = \int_{-\infty}^{+\infty} q\, f_v(q) dq$$

- **Variance**

$$\mathrm{var}(v) = \int_{-\infty}^{+\infty} \left[q - \mathrm{E}(v)\right]^2 \; f_v(q)\, dq$$

- **Standard deviation**

$$\sigma(v) = \sqrt{\mathrm{var}(v)}$$

Tchebicev inequality

$$\mathcal{P}\left(|v - \mathrm{E}(v)| > \epsilon\right) \leq \frac{\mathrm{var}(v)}{\epsilon^2} \qquad \forall \epsilon > 0$$

Random variables (cont.)

Sum of random variables

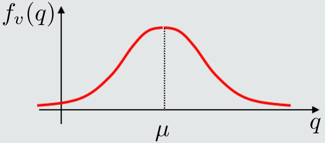
Caution! Given two random variables $v_1(s)$, $v_2(s)$:

$$v(s) = v_1(s) + v_2(s) \quad \Longrightarrow \quad \begin{array}{l} \mathrm{E}(v) = \mathrm{E}(v_1) + \mathrm{E}(v_2) \\ \mathrm{var}(v) \neq \mathrm{var}(v_1) + \mathrm{var}(v_2) \end{array}$$

Important specific case: Gaussian random variable

A r.v. v is Gaussian if:

$$\begin{aligned} f_v(q) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(q-\mu)^2}{2\sigma^2}} \\ \mu &= \mathrm{E}(v) \\ \sigma^2 &= \mathrm{var}(v) \end{aligned}$$



Vector random variable

- For example, given two random variables v_1, v_2 we can build a **random vector** in the obvious way:

$$v = \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix}$$

- Consequently, expectation and variance of a random vector are

$$\mathrm{E}(v) \quad = \quad \begin{bmatrix} \mathrm{E}(v_1) \\ \\ \mathrm{E}(v_2) \end{bmatrix}$$

$$\mathrm{var}(v) \quad = \quad \mathrm{E}\left\{ \left[v - \mathrm{E}(v) \right] \left[v - \mathrm{E}(v) \right]^{\mathrm{T}} \right\}$$

Please note: $\mathrm{var}(v)$ is a matrix!

Vector random variable (cont.)

- In two dimensions

$$v \quad = \quad \begin{bmatrix} v_1 \\ \\ v_2 \end{bmatrix} \qquad \mu_1 = \mathrm{E}(v_1)\ , \quad \mu_2 = \mathrm{E}(v_2)$$

- Therefore

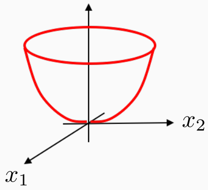
$$\begin{aligned}
\mathrm{var}(v) &= \mathrm{E}\left\{[v - \mathrm{E}(v)]\,[v - \mathrm{E}(v)]^{\mathrm{T}}\right\} \\
&= \mathrm{E}\left\{\begin{bmatrix} v_1 - \mu_1 \\ v_2 - \mu_2 \end{bmatrix} \begin{bmatrix} v_1 - \mu_1 & v_2 - \mu_2 \end{bmatrix}\right\} \\
&= \mathrm{E}\begin{bmatrix} (v_1 - \mu_1)^2 & (v_1 - \mu_1)(v_2 - \mu_2) \\ (v_2 - \mu_2)(v_1 - \mu_1) & (v_2 - \mu_2)^2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 & \sigma_{1\,2} \\ \sigma_{2\,1} & \sigma_2^2 \end{bmatrix} = \Sigma
\end{aligned}$$

variance matrix

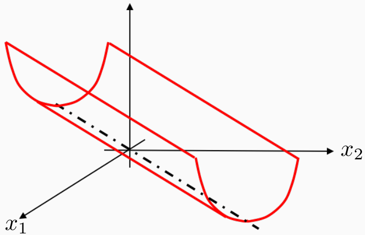
covariance

Vector random variable (cont.)

- The matrix $\Sigma = \mathrm{var}(v)$ in general is symmetric and positive semidefinite



$$x^T \Sigma x \,, \quad \Sigma > 0$$



$$x^T \Sigma x \,, \quad \Sigma \geq 0$$

Correlation and independence

- Two random variables v_1, v_2 are uncorrelated if

$$\mathrm{E}\left\{[v_1 - \mathrm{E}(v_1)]\,[v_2 - \mathrm{E}(v_2)]\right\} = 0$$

that is $\mathrm{E}\,(v_1\, v_2) = \mathrm{E}(v_1) \cdot \mathrm{E}(v_2)$

- Two random variables v_1, v_2 are independent if

$$f_{v_1,\, v_2}\,(a,\, b) = f_{v_1}\,(a) \cdot f_{v_2}\,(b)$$

Independence vs correlation

r.v. independent → r.v. uncorrelated

Stochastic processes

A **stochastic process** is a random phenomenon evolving over time according to a probabilistic law.

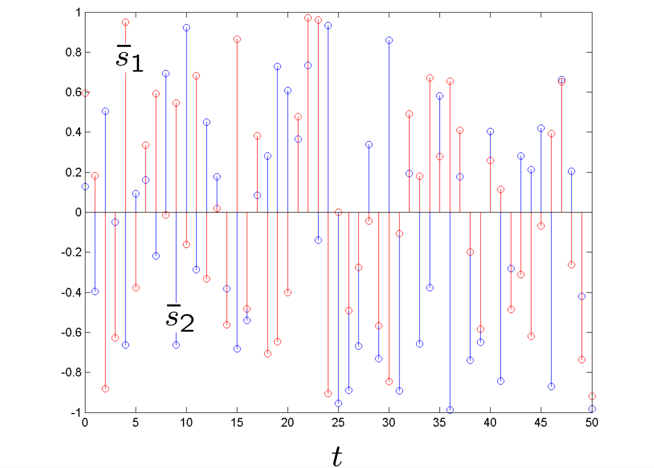
In practice: a two-variable function $v(t,s)$, where t is the time and s is the instance of the random experiment associated with the stochastic process.

Hence

- given $t = \bar{t}$, $v\left(\bar{t}, s\right)$ is a r.v. with a certain probability distribution
- given $\bar{s}$, $v\left(t, \bar{s}\right)$ is a function of time that takes on the name of **realization** of the stochastic process

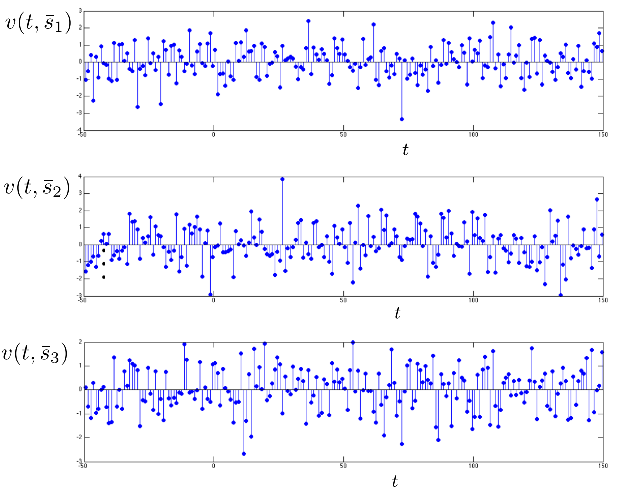
Stochastic processes (cont.)

In practice a stochastic process is a set of infinite r.v. ordered with respect to time.



Stochastic processes (cont.)

In practice a stochastic process is a set of infinite r.v. ordered with respect to time.



Description of a stochastic process

- From a formal point of view, the full description of a stochastic process entails the knowledge of the probability distribution function:

  $$\mathcal{P}\left[x(t_1) \leq x_1\,,\; x(t_2) \leq x_2\,,\; \cdots\,,\; x(t_k) \leq x_k\right]$$

  for every arbitrary value of

  $$k\,,\; x_1\,,\; x_2\,,\; \cdots\,,\; x_k\,,\; t_1\,,\; t_2\,,\; \cdots\,,\; t_k$$

- Such description is clearly not practical. Therefore, we assume that the stochastic process is fully described by the first- and second-order moments.

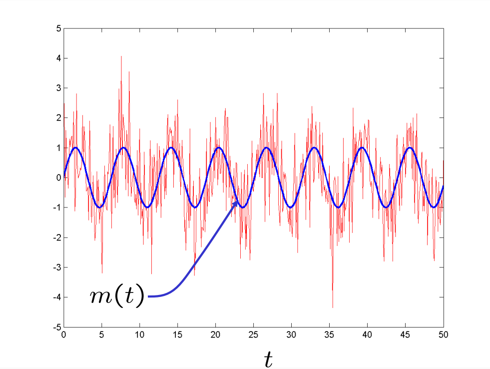
Description of a stochastic process (cont.)

- First-order moment (**expected value** or **average**):

function of t

$$\boxed{m(t)} = \mathrm{E}\left[\boxed{v_t(s)}\right]$$

function of s for given t



Description of a stochastic process (cont.)

- Second-order moment (**covariance function**):

$$\gamma\left(t_1,\, t_2\right) = \mathrm{E}\left\{\left[\underbrace{v_{t_1}(s)}_{\text{function of } s \text{ for given } t_1} - \overbrace{m(t_1)}^{\text{function of } t_1}\right]\left[\underbrace{v_{t_2}(s)}_{\text{function of } s \text{ for given } t_2} - \overbrace{m(t_2)}^{\text{function of } t_2}\right]\right\}$$

- **Correlation function**:

$$\mathrm{E}\left[v_{t_1}(s) \cdot v_{t_2}(s)\right]$$

Coincides with covariance function when $m(t) \equiv 0\ \forall t$.

Description of a stochastic process (cont.)

Therefore:

For our purposes, we assume that a stochastic process is fully described by first- and second-order moments: $m(t)$, $\gamma\left(t_1,\, t_2\right)$.

Two stochastic processes with the same first- and second-order moments are **undistinguishable by hypothesis**.

Stationary stochastic processes

Stationary stochastic process

A stochastic process is stationary (in weak sense) if:

- $m(t) \equiv m =$ const
- $\gamma(t_1, t_2) = \gamma(\tau)\,, \quad \tau = t_2 - t_1$

This assumption greatly simplifies several derivations and, especially, implies the possibility of analyzing the probability distribution without caring about the specific time-instant.

Gaussian stochastic processes

Gaussian processes

irrespective of the choice of the time-instants $t_1,\, t_2,\, \ldots,\, t_N$ the random variables $v_{t_1}(s),\, v_{t_2}(s),\, \ldots,\, v_{t_N}(s)$ are jointly Gaussian, that is:

$$f\left(v_1,\, v_2,\, \ldots,\, v_N\right) = \alpha\, \text{exp} \left\{ -\frac{1}{2} \left(v - \mu\right)^T \Sigma^{-1} \left(v - \mu\right) \right\}$$

where

$$v = \left[v_1,\, v_2,\, \ldots,\, v_N\right]^T \qquad \mu = \mathrm{E}(v) \qquad \Sigma = \mathrm{var}(v)$$

Stationary stochastic process: normalized covariance

- Consider a stationary stochastic process for which:
  - $m(t) \equiv m =$ const
  - $\gamma(t_1, t_2) = \gamma(\tau)\,, \quad \tau = t_2 - t_1$

Clearly, the variance of the process is $\gamma(0)$ and we define the **normalized covariance**:

$$\rho\left(\tau\right) = \frac{\gamma\left(\tau\right)}{\gamma\left(0\right)}$$

White stochastic processes

White process

A stochastic process $\varepsilon(t)$ is defined **white** if

- $\mathrm{E}\left[\varepsilon(t)\right] = 0$
- $\gamma(\tau) = \begin{cases} \lambda^2\,, & \tau = 0 \\ 0\,, & \tau \neq 0 \end{cases}$

and we denote: $\varepsilon \sim \text{WN}\left(0,\, \lambda^2\right)$

In a white process what happens at different time-instants is unrelated, thus the knowledge of $\varepsilon(t)$ does not help in gaining knowledge about $\varepsilon(t+1)$.

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