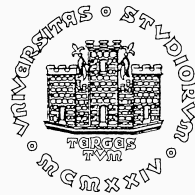


# Systems Dynamics

Course ID: 267MI – Fall 2018

Thomas Parisini  
Gianfranco Fenu

University of Trieste  
Department of Engineering and Architecture



## 267MI –Fall 2018

### Lecture 5

## A (very) short glimpse on probability theory, random variables and discrete-time stochastic processes

### Random variables

- **Random experiment:** analysis of characteristic elements of phenomena yielding unpredictable results.
- **Results space:** we denote by  $S$  the set of all possible results of the experiment. Result:  $s \in S$ .
- **Events:** sets of results of specific interest. Hence an event is a subset of  $S$ .

#### Random variable

Given a random experiment, a **random variable** (r.v.) is a variable  $v(s)$  taking values depending on the result  $s \in S$  of a random experiment via a function  $\varphi(\cdot)$ .

### Probability distribution & density functions

#### Probability distribution function

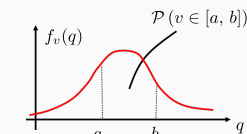
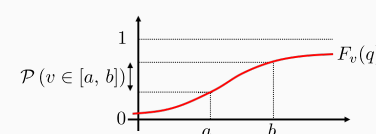
Provides information on the random variable  $v$  and it is defined as

$$F_v(q) = \mathcal{P}(v \leq q)$$

According to the definition  $\mathcal{P}(v \in [a, b]) = F_v(b) - F_v(a)$

#### Probability density function

$$f_v(q) = \frac{dF_v}{dq}$$



Clearly  $\mathcal{P}(v \in [a, b])$  is the **area** “under” the diagram of  $f(q)$  in the interval  $[a, b]$ .

## Functions of random variables

- **Expected value (average value, average)**

$$E(v) = \int_{-\infty}^{+\infty} q f_v(q) dq$$

- **Variance**

$$\text{var}(v) = \int_{-\infty}^{+\infty} [q - E(v)]^2 f_v(q) dq$$

- **Standard deviation**

$$\sigma(v) = \sqrt{\text{var}(v)}$$

### Tchebicev inequality

$$\mathcal{P}(|v - E(v)| > \epsilon) \leq \frac{\text{var}(v)}{\epsilon^2} \quad \forall \epsilon > 0$$

## Random variables (cont.)

### Sum of random variables

**Caution!** Given two random variables  $v_1(s), v_2(s)$ :

$$v(s) = v_1(s) + v_2(s) \implies \begin{aligned} E(v) &= E(v_1) + E(v_2) \\ \text{var}(v) &\neq \text{var}(v_1) + \text{var}(v_2) \end{aligned}$$

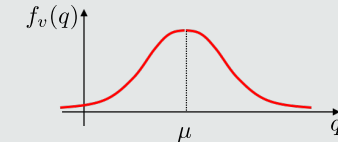
### Important specific case: Gaussian random variable

A r.v.  $v$  is Gaussian if:

$$f_v(q) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(q-\mu)^2}{2\sigma^2}}$$

$$\mu = E(v)$$

$$\sigma^2 = \text{var}(v)$$



## Vector random variable

- For example, given two random variables  $v_1, v_2$  we can build a **random vector** in the obvious way:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- Consequently, expectation and variance of a random vector are

$$E(v) = \begin{bmatrix} E(v_1) \\ E(v_2) \end{bmatrix}$$

$$\text{var}(v) = E\{[v - E(v)][v - E(v)]^T\}$$

Please note:  $\text{var}(v)$  is a **matrix!**

## Vector random variable (cont.)

- In two dimensions

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \mu_1 = E(v_1), \quad \mu_2 = E(v_2)$$

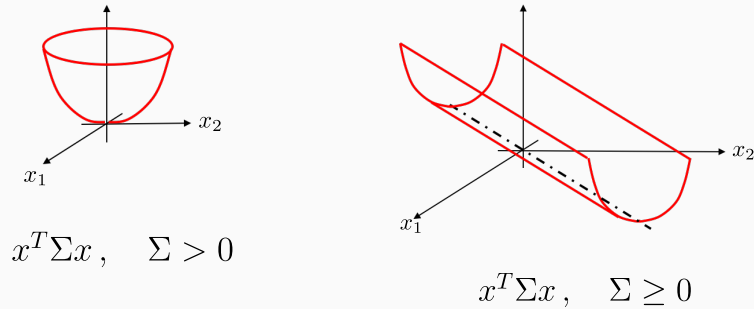
- Therefore

$$\begin{aligned} \text{var}(v) &= E\{[v - E(v)][v - E(v)]^T\} \\ &= E\left\{ \begin{bmatrix} v_1 - \mu_1 \\ v_2 - \mu_2 \end{bmatrix} \begin{bmatrix} v_1 - \mu_1 & v_2 - \mu_2 \end{bmatrix} \right\} \\ &= E \begin{bmatrix} (v_1 - \mu_1)^2 & (v_1 - \mu_1)(v_2 - \mu_2) \\ (v_2 - \mu_2)(v_1 - \mu_1) & (v_2 - \mu_2)^2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \Sigma \quad \text{variance matrix} \end{aligned}$$

covariance

## Vector random variable (cont.)

- The matrix  $\Sigma = \text{var}(v)$  in general is symmetric and positive semidefinite



## Correlation and independence

- Two random variables  $v_1, v_2$  are **uncorrelated** if

$$E \{ [v_1 - E(v_1)] [v_2 - E(v_2)] \} = 0$$

that is  $E(v_1 v_2) = E(v_1) \cdot E(v_2)$

- Two random variables  $v_1, v_2$  are **independent** if

$$f_{v_1, v_2}(a, b) = f_{v_1}(a) \cdot f_{v_2}(b)$$

### Independence vs correlation

r.v. independent  $\longrightarrow$  r.v. uncorrelated

## Stochastic processes

A **stochastic process** is a random phenomenon evolving **over time** according to a probabilistic law.

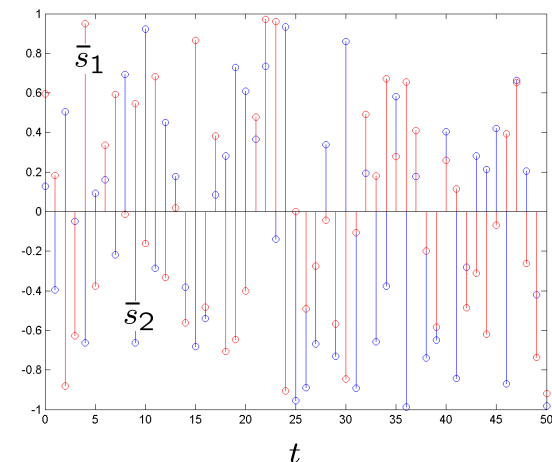
In practice: a two-variable function  $v(t, s)$ , where  $t$  is the time and  $s$  is the instance of the random experiment associated with the stochastic process.

Hence

- given  $t = \bar{t}$ ,  $v(\bar{t}, s)$  is a r.v. with a certain probability distribution
- given  $\bar{s}$ ,  $v(t, \bar{s})$  is a function of time that takes on the name of **realization** of the stochastic process

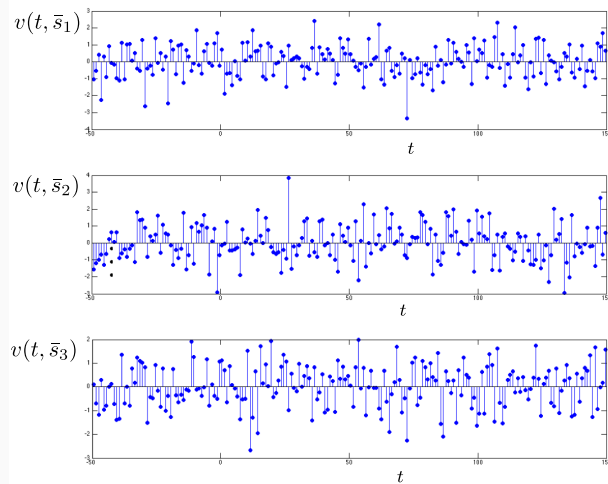
## Stochastic processes (cont.)

In practice a stochastic process is a set of infinite r.v. **ordered with respect to time**.



## Stochastic processes (cont.)

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## Description of a stochastic process

- From a formal point of view, the full description of a stochastic process entails the knowledge of the probability distribution function:

$$\mathcal{P}[x(t_1) \leq x_1, x(t_2) \leq x_2, \dots, x(t_k) \leq x_k]$$

for every arbitrary value of

$$k, x_1, x_2, \dots, x_k, t_1, t_2, \dots, t_k$$

- Such description is clearly not practical. Therefore, we assume that the stochastic process is fully described by the **first- and second-order moments**.

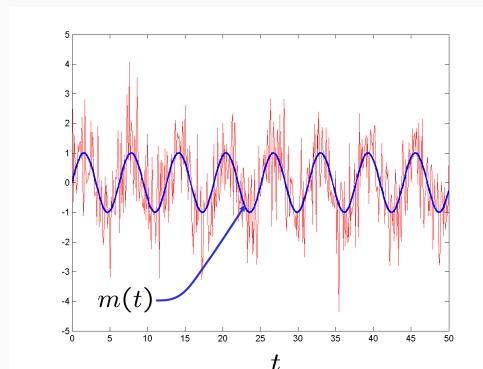
## Description of a stochastic process (cont.)

- First-order moment (**expected value** or **average**):

function of  $t$

$$m(t) = E[v_t(s)]$$

function of  $s$  for given  $t$



## Description of a stochastic process (cont.)

- Second-order moment (**covariance function**):

$$\gamma(t_1, t_2) = E \left\{ \left[ v_{t_1}(s) - m(t_1) \right] \left[ v_{t_2}(s) - m(t_2) \right] \right\}$$

function of  $t_1$

function of  $t_2$

function of  $s$  for given  $t_1$

function of  $s$  for given  $t_2$

- Correlation function:**

$$E[v_{t_1}(s) \cdot v_{t_2}(s)]$$

Coincides with covariance function when  $m(t) \equiv 0 \forall t$ .

## Description of a stochastic process (cont.)

Therefore:

For our purposes, we assume that a stochastic process is fully described by first- and second-order moments:  $m(t)$ ,  $\gamma(t_1, t_2)$ .



Two stochastic processes with the same first- and second-order moments are **undistinguishable by hypothesis**.

## Stationary stochastic processes

### Stationary stochastic process

A stochastic process is stationary (in weak sense) if:

- $m(t) \equiv m = \text{const}$
- $\gamma(t_1, t_2) = \gamma(\tau)$ ,  $\tau = t_2 - t_1$

This assumption greatly simplifies several derivations and, especially, implies the possibility of analyzing the probability distribution without caring about the specific time-instant.

## Gaussian stochastic processes

### Gaussian processes

irrespective of the choice of the time-instants  $t_1, t_2, \dots, t_N$  the random variables  $v_{t_1}(s), v_{t_2}(s), \dots, v_{t_N}(s)$  are jointly Gaussian, that is:

$$f(v_1, v_2, \dots, v_N) = \alpha \exp \left\{ -\frac{1}{2} (v - \mu)^T \Sigma^{-1} (v - \mu) \right\}$$

where

$$v = [v_1, v_2, \dots, v_N]^T \quad \mu = E(v) \quad \Sigma = \text{var}(v)$$

## Stationary stochastic process: normalized covariance

- Consider a stationary stochastic process for which:
  - $m(t) \equiv m = \text{const}$
  - $\gamma(t_1, t_2) = \gamma(\tau)$ ,  $\tau = t_2 - t_1$

Clearly, the **variance of the process** is  $\gamma(0)$  and we define the **normalized covariance**:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}$$

### White process

A stochastic process  $\varepsilon(t)$  is defined **white** if

- $E[\varepsilon(t)] = 0$
- $\gamma(\tau) = \begin{cases} \lambda^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$

and we denote:  $\varepsilon \sim \text{WN}(0, \lambda^2)$

In a white process what happens at different time-instants is unrelated, thus the knowledge of  $\varepsilon(t)$  does not help in gaining knowledge about  $\varepsilon(t+1)$ .

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END