Systems Dynamics

Course ID: 267MI – Fall 2018

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Lecture 7
Dynamic models of stationary discrete-time stochastic processes

Signal energy & power

Signal Energy and Power

- The terms **signal energy** and **signal power** are used to characterize a continuous-time or discrete-time signal.
- They are not actually energy and power measurements, even though the energy and power signal definitions are inspired by expressions used to evaluate energy or power of electrical signals.
- Indeed, the definition of signal energy and power refers to any signal, including signals that take on complex values.

Energy and power for continuous-time signals

- Consider a generic *deterministic* continuous-time signal $x(t)$
- Let's define
  - instantaneous power
$$P(t) = |x(t)|^2$$
  - energy over the time period $[t_0\,,\ t_1]$
$$E\,(t_0\,,\ t_1) = \int_{t_0}^{t_1} |x(t)|^2\,d\,t$$
  - average power over the time period $[t_0\,,\ t_1]$
$$P\,(t_0\,,\ t_1) = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} |x(t)|^2\,d\,t$$

Analogy

Consider a current signal $i(t)$ flowing through a transmission line represented by resistance $R = 1\Omega$. Evaluate the energy loss in the line, the instantaneous and average power loss in the line and compare the results with the definitions above.

Energy and power for continuous-time signals (cont.)

- Extending the time period, let's define
  - **energy** of a deterministic signal $x(t)$

$$E_\infty = \int_{-\infty}^{+\infty} |x(t)|^2 \, d\,t$$

  - **power** of a deterministic signal $x(t)$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 \, d\,t$$

Energy and power for a continuous-time deterministic signal

- the deterministic continuous-time signal $x(t)$ is called an **energy signal** if $0 < E_\infty < \infty$
- the deterministic continuous-time signal $x(t)$ is called a **power signal** if $0 < P_\infty < \infty$

Energy and power for discrete-time signals

- Consider a generic discrete-time *deterministic* signal $\{x(k)\}$
- Let's define
  - instantaneous power
    $$P(k) = |x(k)|^2$$
  - energy over the time period $[k_0\ ,\ k_1]$
    $$E\left(k_0\ ,\ k_1\right) = \sum_{k=k_0}^{k_1} |x(k)|^2$$
  - average power over the time period $[k_0\ ,\ k_1]$
    $$P\left(k_0\ ,\ k_1\right) = \frac{1}{k_1 - k_0 + 1} \sum_{k=k_0}^{k_1} |x(k)|^2$$

Energy and power for discrete-time signals (cont.)

Analogously to the continuous-time case, let's define

- **energy** of a discrete-time signal $x(k)$

$$E_\infty = \sum_{-\infty}^{+\infty} |x(k)|^2$$

- **power** of a discrete-time signal $x(k)$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{+N} |x(k)|^2$$

Energy and power for a discrete-time deterministic signal

- the discrete-time signal $x(k)$ is called an **energy signal** if $0 < E_\infty < \infty$
- the discrete-time signal $x(k)$ is called a **power signal** if $0 < P_\infty < \infty$

Energy and power for stationary stochastic processes

- Consider now a discrete-time ***stationary stochastic process (in weak sense)***. Let's define
  - instantaneous power

$$P(k) = \mathrm{E}\left[|x(k)|^2\right]$$

  - **power** of a stationary (in weak sense) stochastic process $x(k)$

$$P_\infty = \mathrm{E}\left\{\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{+N} |x(k)|^2\right\} = \mathrm{E}\left\{|x(k)|^2\right\}$$

Energy and power for a discrete-time stationary (in weak sense) stochastic process

- $E_\infty \to \infty$
- $P_\infty = \mathrm{var}(x) + \mathrm{E}\left[(x(k))^2\right]$

Signal Energy and Power

Remarks

- The power for an energy signal is zero

$$E_\infty < \infty \quad \Longrightarrow \quad P_\infty = 0$$

- The energy for a power signal is infinite

$$P_\infty < \infty \quad \Longrightarrow \quad E_\infty \to \infty$$

- Some signals are neither energy nor power signals.
- A signal can't be both an energy signal and a power signal.

Examples

- Pure deterministic signals

$$x(k) = A\,,\ k = 1\,,\ 2\,,\ \dots \quad \Longrightarrow \quad E_\infty \to \infty\,,\ P_\infty = A^2$$

$$v(k) = A\,e^{-k}\,,\ k = 1\,,\ 2\,,\ \dots \quad \Longrightarrow \quad E_\infty = \sum_{k=1}^{+\infty} A^2\,e^{-2k}\,,\ P_\infty = 0$$

- Pure stochastic signal

$$\eta(k) = \mathsf{WN}\left(0\,,\ \lambda^2\right) \quad \Longrightarrow E_\infty \to \infty\,,\ P_\infty = \lambda^2$$

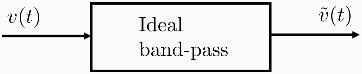
Spectral representation of stationary stochastic processes

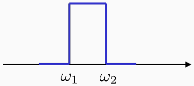
Description of stationary stochastic processes

- We described a zero-mean stationary process $v(t)$ by the correlation function:

$$\gamma(\tau) = E[v(t)v(t+\tau)]$$

- However, it is of customary importance to devise a frequency-based description of stationary stochastic processes
- Consider the ideal conceptual scheme:





$\mathrm{var}\,[v(t)] = \lambda^2 \longrightarrow \mathrm{var}\,[\tilde{v}(t)] = \tilde{\lambda}^2$

Average power of the process

Power contribution in the frequency-interval $[\omega_1\,,\ \omega_2]$

Spectral power density (spectrum)

- Thus, we introduce the spectral power density (spectrum) as the Fourier transform of the correlation function:

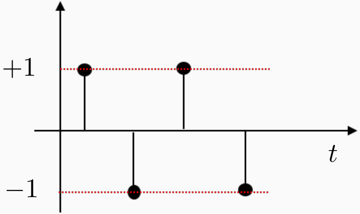
$$\Gamma(\omega) = \mathcal{F}\{\gamma(\tau)\} = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) e^{-j\omega\tau} \quad (\star)$$

In order the Fourier transform $(\star)$ to converge, it is necessary that $\gamma(\tau) \to 0$ as $\tau \to \infty$ "quickly enough", that is:

$$\sum_{\tau=-\infty}^{+\infty} |\gamma(\tau)| < \infty \quad \longrightarrow \quad \begin{array}{l} \Gamma(\omega) \quad \text{exists} \quad \forall\omega \\ \\ \Gamma(\omega) \quad \text{is continuous} \end{array}$$

Spectral power density (spectrum) (cont.)

- Consider the discrete-time periodic sequence:



This sequence is characterized by the maximum possible frequency of variation (sign-change at every time-instant). The smallest period is $T = 2$ and hence the maximum frequency is $1/2$. Consequently the maximum angular frequency is $\dfrac{2\pi}{T} = \pi$.

$\boxed{\Gamma \text{ is evaluated between } -\pi \text{ and } \pi}$

Properties

- $\Gamma(\omega) \in \mathbb{R}$
  In fact $\gamma(\tau) = \gamma(-\tau)$ (correlation function is even)

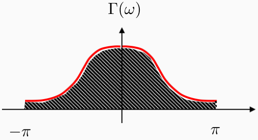
$$e^{-j\omega(-\tau)}\gamma(-\tau) + e^{-j\omega\tau}\gamma(\tau) = \gamma(\tau)\left[e^{j\omega\tau} + e^{-j\omega\tau}\right] \in \mathbb{R}$$

- $\Gamma(\omega)$ is even, that is $\Gamma(\omega) = \Gamma(-\omega)$
- $\Gamma(\omega) \geq 0$
- $\Gamma(\omega)$ is periodic of period 2π
- $\gamma(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \Gamma(\omega) e^{j\omega\tau} d\omega$

Properties (cont.)

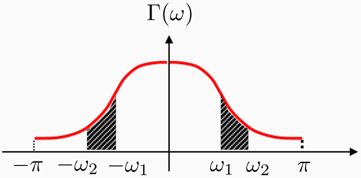
- Thus:

$$\mathrm{var}[v(t)] = \gamma(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma(\omega)d\omega$$



Area "below" $\Gamma(\omega)$ in $[-\pi\ ,\ \pi]$ is proportional to the process power and hence to its "variability".

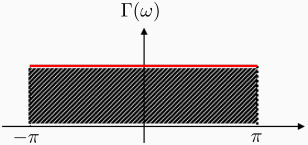
Area below $\Gamma(\omega)$ in $[-\omega_2\ ,\ -\omega_1]$ and $[\omega_1\ ,\ \omega_2]$ represents the distribution of the "process variability" in the angular frequency range $[\omega_1\ ,\ \omega_2]$.



Example

$$v(\cdot) \sim WN(0, \lambda^2)$$

$$\gamma(\tau) = \begin{cases} \lambda^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$



$$\Gamma(\omega) = \frac{1}{2\pi}\lambda^2 \times 2\pi = \lambda^2 \,, \quad \forall \omega \in [-\pi, \pi]$$

Then, in a white process, all angular frequencies contribute in the same way to the overall "process variability"

Dynamic representations of stationary stochastic processes

- The representations seen up to now are static: the process is considered in its entirety (sequence of infinite r.v. from $-\infty$ to $+\infty$).
- These representations, though correct, are not useful for the solution of the prediction problem.
- We need to devise **dynamic** representations in which it is possible **to relate the future evolution of the process with its past**.
- Special care has to be exercised

Example 1

Consider a r.v. v and define the process $v(t) = v$

$v(t, s) = v(s)$ (all realizations have constant behavior)

By observing $v(t, s)$ at some time-instant **the uncertainty disappears** in the sense that the value that the process will take on in the future will not change.

Only **a priori** uncertainty:
BEFORE the "observation" of the process.

Example 2

$v(\cdot) \sim WN(0, \lambda^2)$

Opposite situation with respect to Example 1: the observation of past values of the process does not help in predicting future values (past and future are not correlated).

$\boxed{\text{The best prediction is the expected value } (= 0)}$

Dynamic representations of stochastic processes (cont.)

- The two opposite (and kind of "extreme") examples show that, in the context of the solution of the prediction problem, a more peculiar description of the stochastic process is needed.
- Define a vector space $\mathcal{G}$ and assume to represent the r.v. $v(t), v(t-1), v(t-2), \ldots$ as vectors in $\mathcal{G}$.
- Define
  - $\mathcal{H}_t[v]$ — Subspace of the past with respect to t
    hyper-plane generated by vectors associated with observations $v(t), v(t-1), v(t-2), \ldots$
  - $\tilde{\mathcal{H}}_t[v] = \bigcap_t \mathcal{H}_t[v]$ — Subspace of the "remote" past

Dynamic representations of stochastic processes (cont.)

- $\hat{\mathcal{H}}_t[v] = \mathcal{H}_t[v] \setminus \tilde{\mathcal{H}}_t[v]$ such that $\hat{\mathcal{H}}_t[v] \perp \tilde{\mathcal{H}}_t[v]$ — Orthogonal complement

  Hence, the vectors in $\hat{\mathcal{H}}_t[v]$ are orthogonal to all vectors in $\tilde{\mathcal{H}}_t[v]$
- $\tilde{v}(t)$ projection of $v(t)$ on $\tilde{\mathcal{H}}_t[v]$ — Purely deterministic component
- $\hat{v}(t)$ projection of $v(t)$ on $\hat{\mathcal{H}}_t[v]$ — Purely non-deterministic component

Wold decomposition

$$\tilde{v}(t) \perp \hat{v}(t)$$

Once v is decomposed on the two components $\tilde{v}\,,\ \hat{v}$, only the purely non-deterministic component is useful for solving the prediction problem as the purely deterministic component is perfectly predictable from the past.

Dynamic representations of stochastic processes (cont.)

- Then, from now on, let us refer to dynamic representation of purely non-deterministic processes
- Consider the hyper-plane $\hat{\mathcal{H}}_t[v]$ and consider a basis of orthogonal vectors having the same norm

$$\eta(t), \eta(t-1), \eta(t-2), \ldots$$

- Such basis is chosen in such a way that:
  - $\eta(t-1), \eta(t-2), \eta(t-3), \ldots$ basis for $\hat{\mathcal{H}}_{t-1}[v]$
  - $\eta(t-2), \eta(t-3), \eta(t-4), \ldots$ basis for $\hat{\mathcal{H}}_{t-2}[v]$
  - $\ldots$

  vectors $\eta(t), \eta(t-1), \eta(t-2), \ldots$ correspond to r.v. mutually uncorrelated with the same norm

$$\boxed{\eta(\cdot) \text{ is a white process}}$$

Dynamic representations of stochastic processes (cont.)

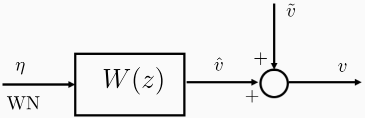
- Project $v(t)$ on $\eta(t), \eta(t-1), \eta(t-2), \ldots$
  - $w(0), w(1), w(2), \ldots$ projections components
  - Write the projection $\hat{v}(t)$ as

$$\hat{v}(t) = w(0)\eta(t) + w(1)\eta(t-1) + \cdots$$

$$= \boxed{\sum_{i=-\infty}^{t} w(t-i)\eta(i)} \qquad \text{convolution of } w(\cdot)\,,\ \eta(\cdot)$$

time-invariance

Hence:



where

$$W(z) = \mathcal{Z}[w(t)] = \sum_{i=0}^{\infty} w(i) z^{-i}$$

$w(t)$ is the **impulse response** of the system

Dynamic representations of stochastic processes (cont.)

Therefore

The purely non deterministic component of a **stationary** stochastic process can be seen as the output of a discrete-time dynamic systems driven by a white input process.

Remark

$W(z)$ is not necessarily a rational function of polynomials in z.

Dynamic representations of stochastic processes (cont.)

Important

Purely non deterministic and purely deterministic processes are **very different from a spectral point of view**.

Example
Consider the process:

$$v(t) = a_1 v(t-1) + a_2 v(t-2) + \cdots + a_n v(t-n)$$

Clearly $v(t)$ is a a **known** linear combination of past values $v(t-1)\,,\ v(t-2)\,,\ \ldots\,,\ v(t-n)$ hence, a purely deterministic process.

Dynamic representations of stochastic processes (cont.)

Example (cont.)
Now

$$P(z)\,v(t) = 0 \qquad P(z) = 1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}$$

This is possible only if $v(t) = \alpha_1 \lambda_1^t + \cdots \alpha_n \lambda_n^t$ where $\lambda_1\,,\ \ldots\,,\ \lambda_n$ are the zeros of

$$P(z) = 1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n} = \frac{z^n - a_1 z^{n-1} - \cdots - a_n}{z^n}$$

Remark

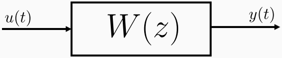
In general: the spectral content of purely deterministic processes is “concentrated” in a **finite** number of **discrete** frequency points.

Instead, the spectral content of purely non-deterministic processes is spread over the entire frequency spectrum.

Analysis of dynamic systems driven by input stationary stochastic processes

Properties of LTI discrete-time systems

- Consider a linear time-invariant dynamic system with $W(z)$ as transfer function:



where

$$W(z) = \frac{\gamma_0 z^n + \gamma_1 z^{n-1} + \cdots \gamma_n}{\alpha_0 z^n + \alpha_1 z^{n-1} + \cdots \alpha_n}$$

- Hence

$$\alpha_0 y(t) + \alpha_1 y(t-1) + \cdots + \alpha_n y(t-n) =$$
$$= \gamma_0 u(t) + \gamma_1 u(t-1) + \cdots + \gamma_n u(t-n)$$

Properties of LTI discrete-time systems (cont.)

- Recall that, if all poles of $W(z)$ are located strictly inside the unit circle and if

$$u(t) = A \sin\left(\frac{2\pi}{\bar{T}} t + \beta\right), \quad t = 0, 1, \ldots$$

then a sinusoidal regime takes place after transient

$$y(t) \simeq A \left| \, W(e^{j\bar{\omega}}) \, \right| \sin\left(\frac{2\pi}{\bar{T}} t + \beta + \arg\left[\, W(e^{j\bar{\omega}}) \, \right]\right)$$

Evaluation of $W(z)$
on the unit circle

where

$$\bar{\omega} = \frac{2\pi}{\bar{T}}$$

Properties of LTI discrete-time systems (cont.)

- Due to linearity, it follows that

$$y(t) = \underbrace{y_F(t)}_{\text{Forced response (zero initial conditions)}} + \underbrace{y_L(t)}_{\text{Free response (zero input)}}$$

- Then

$$y_F(t) = \sum_{j=-t_0}^{t} w(t-j)u(j)$$

  where $w(t)$ is the impulse response of the system

- Due to the assumed asymptotic stability, we have

$$\lim_{t_0 \to -\infty} y_L(t) = 0$$

  hence

$$\lim_{t_0 \to -\infty} y(t) = \sum_{j=-\infty}^{t} w(t-j)u(j)$$

Properties of LTI discrete-time systems (cont.)

Then

If all poles of $W(z)$ are located strictly inside the unit circle, irrespective of the initial conditions, for $t_0 \to -\infty$ we have

$$y(t) = \sum_{j=-\infty}^{t} w(t-j)u(j) = \sum_{i=0}^{+\infty} w(i)u(t-i)$$

Convolution formula

A stochastic process as input of an LTI system

- Now, assume that $u(\cdot)$ is a stochastic process with zero expected value (that is $\mathrm{E}(u) = 0$)
- Thus

$$
\begin{aligned}
\mathrm{E}[y(t)] &= \mathrm{E}\left[\sum_{i=0}^{\infty} w(i)u(t-i)\right] \\
&= \sum_{i=0}^{\infty} w(i)\,\mathrm{E}\left[u(t-i)\right] \\
&= \sum_{i=0}^{\infty} w(i)\,\boxed{\mathrm{E}\left[u(t)\right]} \\
&= 0
\end{aligned}
$$

stationary

A stochastic process as input of an LTI system (cont.)

- As we established that $y(\cdot)$ has zero expected value the covariance functions coincide with the correlation functions.
- Consider input and output values at different time instant t_1 and t_2:

$$y(t_2) = \sum_{i=0}^{\infty} w(i) u(t_2 - i)$$

$$u(t_1) y(t_2) = \sum_{i=0}^{\infty} w(i) u(t_1) u(t_2 - i)$$

$$y(t_1) y(t_2) = \sum_{i=0}^{\infty} w(i) y(t_1) u(t_2 - i)$$

A stochastic process as input of an LTI system (cont.)

- Hence, evaluating the expected values

$$\mathrm{E}\left[u(t_1)y(t_2)\right] = \sum_{i=0}^{\infty} w(i)\,\mathrm{E}\left[u(t_1)u(t_2-i)\right]$$

$$\boxed{\gamma_{uy}(t_1,t_2) = \sum_{i=0}^{\infty} w(i)\gamma_{uu}(t_1,t_2-i)}$$

$$\mathrm{E}\left[y(t_1)y(t_2)\right] = \sum_{i=0}^{\infty} w(i)\,\mathrm{E}\left[y(t_1)u(t_2-i)\right]$$

$$\boxed{\gamma_{yy}(t_1,t_2) = \sum_{i=0}^{\infty} w(i)\gamma_{yu}(t_1,t_2-i)}$$

A stochastic process as input of an LTI system (cont.)

- As we assumed that the process $u(\cdot)$ is stationary, we obtain:

$$\boxed{\gamma_{uy}(\tau) = \sum_{i=0}^{\infty} w(i)\gamma_{uu}(\tau - i)}$$

$$\boxed{\gamma_{yy}(\tau) = \sum_{i=0}^{\infty} w(i)\gamma_{yu}(\tau - i)}$$

  - Then, the correlation function between the input and output processes is the convolution of the impulse response with the auto-correlation function of the input.
  - Analogously, the auto-correlation function of the output process is given by the convolution of the impulse response with the input-output correlation function.

Correlation functions and complex spectra

- Now, introduce the two-sided Z transform of all the possible correlation functions:

$$\Phi_{uu}(z) = \sum_{\tau=-\infty}^{+\infty} \gamma_{uu}(\tau) z^{-\tau}$$

$$\Phi_{yy}(z) = \sum_{\tau=-\infty}^{+\infty} \gamma_{yy}(\tau) z^{-\tau}$$

$$\Phi_{uy}(z) = \sum_{\tau=-\infty}^{+\infty} \gamma_{uy}(\tau) z^{-\tau}$$

$$\Phi_{yu}(z) = \sum_{\tau=-\infty}^{+\infty} \gamma_{yu}(\tau) z^{-\tau}$$

Correlation functions and complex spectra (cont.)

- Recalling the definitions of spectral power density, we have:

$$\Gamma_{uu}(\omega) = \left.\Phi_{uu}(z)\right|_{z=e^{j\omega}} = \Phi_{uu}\left(e^{j\omega}\right)$$

$$\Gamma_{yy}(\omega) = \left.\Phi_{yy}(z)\right|_{z=e^{j\omega}} = \Phi_{yy}\left(e^{j\omega}\right)$$

where $\Phi_{uu}(z)$ and $\Phi_{yy}(z)$ take on the name of **complex spectra** of the input and the output.

Properties

- $\Gamma_{uu}(\omega),\ \Gamma_{yy}(\omega) \in \mathbb{R}$
  - though in general $\Gamma_{uy}(\omega),\ \Gamma_{yu}(\omega) \in \mathbb{C}$

- $\gamma_{uy}(\tau) = \gamma_{yu}(-\tau)$

$$\Phi_{yu}(z) = \Phi_{uy}(z^{-1})$$

$$\Gamma_{yu}(\omega) = \Gamma^*_{uy}(\omega)$$

Properties (cont.)

- Moreover

$$\gamma_{uy}(\tau) = \sum_{i=0}^{\infty} w(i)\gamma_{uu}(\tau - i) \qquad \Phi_{uy}(z) = W(z)\Phi_{uu}(z)$$

$$\gamma_{yy}(\tau) = \sum_{i=0}^{\infty} w(i)\gamma_{yu}(\tau - i) \qquad \Phi_{yy}(z) = W(z)\Phi_{yu}(z)$$

But

$$\Phi_{yu}(z) = \Phi_{uy}(z^{-1}) = W(z^{-1})\Phi_{uu}(z^{-1})$$

$$\Phi_{yy}(z) = W(z)W(z^{-1})\Phi_{uu}(z^{-1})$$

$$\boxed{\Phi_{yy}(z) = W(z)W(z^{-1})\Phi_{uu}(z)}$$

This is a very important result: we are able to compute the complex spectrum of the output process of a LTI asymptotically stable system driven by a stationary input process.

Properties (cont.)

- Going back to the frequency domain:

$$\Gamma_{yy}(\omega) = \Phi_{yy}(e^{j\omega}) = W(e^{j\omega})W(e^{-j\omega})\Phi_{uu}(e^{j\omega})$$
$$= W(e^{j\omega})W(e^{-j\omega})\Gamma_{uu}(\omega)$$

$$\boxed{\Gamma_{yy}(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_{uu}(\omega)}$$

This is a very important result as well: we relate spectral power densities with the frequency response.

Stationary processes with rational spectra

More common stationary processes with rational spectra

- White noise
- Moving average process (**MA**)
- Auto-regressive process (**AR**)
- Auto-regressive and Moving average process (**ARMA**)

Common characteristic: as we will see, these processes are generated from a filtered white process.

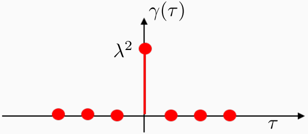
Analysis of dynamic systems driven by input stationary stochastic processes

White noise

White noise

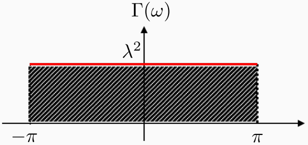
$$v(\cdot) \sim WN(0, \lambda^2)$$

$$\gamma(\tau) = \begin{cases} \lambda^2, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$



$$\Gamma(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) e^{-j\omega\tau}$$

$$= \gamma(0) = \lambda^2$$



As can be seen from the correlation function, in a white process the values of the process in different time-instants do not have any mutual relation, that is, the knowledge of $v(t)$ is of no help to estimate $v(\bar{t})$, $\bar{t} \neq t$.

Analysis of dynamic systems driven by input stationary stochastic processes

Moving average process (MA)

MA process

- Given a white process $\eta(\cdot) \sim WN(0, \lambda^2)$
- A MA process of order n (denoted as MA(n)) is the process

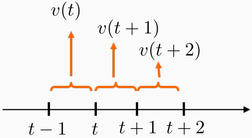
$$v(t) = c_0\eta(t) + c_1\eta(t-1) + c_2\eta(t-2) + \cdots + c_n\eta(t-n)$$

- Hence $v(t)$ is a linear combination (average) of the values taken on by the white process in the time-window from $t-n$ to t. When t increases, this time-window shifts (moving).

Example 1

- Consider the process MA(1) :

$$v(t) = c_0\eta(t) + c_1\eta(t-1)$$



- Expected value:

$$\begin{aligned} \mathrm{E}[v(t)] &= \mathrm{E}[c_0\eta(t) + c_1\eta(t-1)] \\ &= c_0\,\mathrm{E}[\eta(t)] + c_1\,\mathrm{E}[\eta(t-1)] = 0 \end{aligned}$$

Example 1 (cont.)

- Correlation function (= covariance due to the zero expected value):

$$\gamma(t_1, t_2) = E[v(t_1)v(t_2)]$$

A priori could be non-stationary

- $t_1 = t_2 = t$

$$\gamma(t,t) = \mathrm{E}\left\{v(t)^2\right\} = \mathrm{E}\left\{[c_0\eta(t) + c_1\eta(t-1)]^2\right\}$$
$$= c_0^2 \underbrace{\mathrm{E}\left\{\eta(t)^2\right\}}_{\lambda^2} + c_1^2 \underbrace{\mathrm{E}\left\{\eta(t-1)^2\right\}}_{\lambda^2} + 2c_0c_1 \underbrace{\mathrm{E}\left\{\eta(t)\eta(t-1)\right\}}_{0}$$
$$= \left(c_0^2 + c_1^2\right)\lambda^2$$

Example 1 (cont.)

- Correlation function (cont.)
  - $t_2 = t_1 \pm 1$

$$\gamma(t, t\pm 1) = \mathrm{E}\left\{v(t)v(t\pm 1)\right\}$$

$$= \mathrm{E}\left\{\left[c_0\eta(t) + c_1\eta(t-1)\right]\left[c_0\eta(t\pm 1) + c_1\eta(t-1\pm 1)\right]\right\}$$

$$= c_0^2 \overbrace{\mathrm{E}\left\{\eta(t)\eta(t\pm 1)\right\}}^{0} + c_1^2 \overbrace{\mathrm{E}\left\{\eta(t-1)\eta(t-1\pm 1)\right\}}^{0}$$

$t_2 = t_1 - 1$: 0 $\qquad$ $t_2 = t_1 - 1$: λ^2

$$+ c_0c_1 \mathrm{E}\left\{\eta(t)\eta(t-1\pm 1)\right\} + c_0c_1 \mathrm{E}\left\{\eta(t-1)\eta(t\pm 1)\right\}$$

$t_2 = t_1 + 1$: λ^2 $\qquad$ $t_2 = t_1 + 1$: 0

$$= c_0c_1\lambda^2 = \gamma(t, t\pm 1)$$

Example 1 (cont.)

- Correlation function (cont.)
  - $t_2 = t_1 \pm 2$
    $$\gamma(t, t \pm 2) = E\left\{v(t)v(t \pm 2)\right\} = 0$$
  - In general:
    $$\gamma(t, t \pm k) = E\left\{v(t)v(t \pm k)\right\} = 0\,, \quad k \geq 2$$
- Note that $\gamma\left(t_1\,,\; t_2\right)$ actually is always only a function of $t_2 - t_1$ and hence the process MA(1) is stationary.

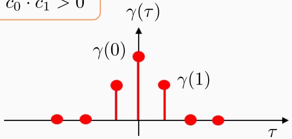
Example 1 (cont.)

- Thus:

$$MA(1): \begin{cases} \gamma(0) = \left(c_0^2 + c_1^2\right) \lambda^2 \\ \gamma(1) = c_0 c_1 \lambda^2 \\ \gamma(k) = 0, \; \forall\, k > 1 \end{cases}$$

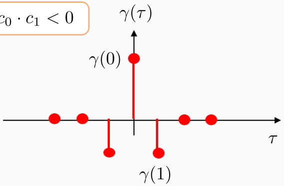
Two possible cases:

$\boxed{c_0 \cdot c_1 > 0}$



tendency not to change sign in consecutive time-instants

$\boxed{c_0 \cdot c_1 < 0}$

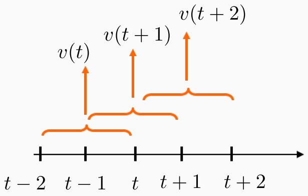


tendency to change sign in consecutive time-instants

Example 2

- Consider the process MA(2) :

$$v(t) = c_0\eta(t) + c_1\eta(t-1) + c_2\eta(t-2)$$



- Expected value:

$$\mathrm{E}[v(t)] = \mathrm{E}[c_0\eta(t) + c_1\eta(t-1) + c_2\eta(t-2)]$$

$$= c_0\,\mathrm{E}[\eta(t)] + c_1\,\mathrm{E}[\eta(t-1)] + c_2\,\mathrm{E}[\eta(t-2)] = 0$$

Example 2 (cont.)

- Correlation function (= covariance due to the zero expected value):
  By a simple algebra analogous to the previous case, we have:

$$\gamma(t,t) = \left(c_0^2 + c_1^2 + c_2^2\right) \lambda^2$$

$$\gamma(t,t \pm 1) = \left(c_0 c_1 + c_1 c_2\right) \lambda^2$$

$$\gamma(t,t \pm 2) = \left(c_0 c_2\right) \lambda^2$$

$$\gamma(t,t \pm k) = 0 \,, \quad \forall k > 2$$

- Here again we notice that $\gamma(t_1\,,\; t_2)$ is always only a function of $t_2 - t_1$ and hence the process MA(2) is stationary.

MA(n) process

- In general, for a process MA(n) we can prove that:

$$MA(n): \quad \gamma(\tau) = \begin{cases} 0 & \text{if} \quad |\tau| > n \\ \left(c_0 c_{|\tau|} + c_1 c_{|\tau|+1} + \cdots + c_{n-|\tau|} c_n\right) \lambda^2 & \text{if} \quad n \geq |\tau| \end{cases}$$

- The stationarity should not be surprising: the process is just a linear combination of values taken on by a stationary process.
- Clearly:

$$\text{var}\left[v(t)\right] = \gamma(0) = \left(c_0^2 + c_1^2 + \cdots + c_n^2\right) \lambda^2$$

MA(n) process (cont.)

- Let us consider the general expression of a process MA(n):

$$v(t) = c_0\eta(t) + c_1\eta(t-1) + c_2\eta(t-2) + \cdots + c_n\eta(t-n)$$

- By using the unity delay operator z^{-1} we have:

$$\begin{aligned} v(t) &= c_0\eta(t) + c_1z^{-1}\eta(t) + c_2z^{-2}\eta(t) + \cdots + c_nz^{-n}\eta(t) \\ &= \left(c_0 + c_1z^{-1} + c_2z^{-2} + \cdots + c_nz^{-n}\right)\eta(t) \\ &= C(z)\eta(t) \end{aligned}$$

  where we set

$$C(z) = c_0 + c_1z^{-1} + c_2z^{-2} + \cdots + c_nz^{-n}$$

- Then, the transfer function turns out to be

$$W(z) = \frac{c_0z^n + c_1z^{n-1} + c_2z^{n-2} + \cdots + c_n}{z^n}$$

  which is **asymptotically stable** (n poles in the origin).

Important remark

- In general, stationary processes are characterized via the expected value and the covariance function
- In the case of the process MA(n) the **expected value** is zero and the **covariance function** is given by

$$MA(n): \quad \gamma(\tau) = \begin{cases} 0 & \text{if} \quad |\tau| > n \\ \left(c_0 c_{|\tau|} + c_1 c_{|\tau|+1} + \cdots + c_{n-|\tau|} c_n\right) \lambda^2 & \text{if} \quad n \geq |\tau| \end{cases}$$

The process is fully characterized by the **parameters**
$c_0, c_1, \ldots, c_n, \lambda^2$

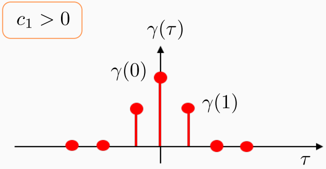
Important remark (cont.)

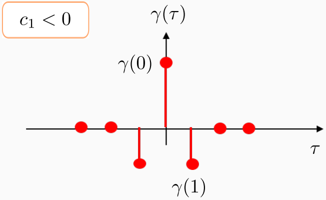
- However: the characterization by the parameters $c_0, c_1, \ldots, c_n, \lambda^2$ is redundant.
  In fact: let
  $$\tilde{c}_0 = \alpha c_0, \tilde{c}_1 = \alpha c_1, \ldots, \tilde{c}_n = \alpha c_n$$
  and consider the process
  $$\tilde{v}(t) = \tilde{c}_0 \tilde{\eta}(t) + \tilde{c}_1 \tilde{\eta}(t-1) + \cdots + \tilde{c}_n \tilde{\eta}(t-n)$$
  where $\tilde{\eta}(\cdot) \sim \mathrm{WN}(0, \tilde{\lambda}^2)$.
  If
  $$\tilde{\lambda}^2 = \frac{\lambda^2}{\alpha^2} \quad \Longrightarrow \quad \tilde{\gamma}(\tau) = \gamma(\tau)$$
  and hence the two processes $v(t)$ $\tilde{v}(t)$ are not distinguishable.
- The redundancy is eliminated assigning one of the parameters. The typical choice is $c_0 = 1$ and then the process MA(n) is written as
  $$v(t) = \eta(t) + c_1 \eta(t-1) + \cdots + c_n \eta(t-n)$$

Example 1 (continued)

$$MA(1): \quad v(t) = \eta(t) + c\,\eta(t-1)$$

$$MA(1): \begin{cases} \gamma(0) = \left(1 + c^2\right)\lambda^2 \\ \gamma(1) = c\,\lambda^2 \\ \gamma(k) = 0,\ \forall\, k > 1 \end{cases}$$





Example 1 (cont.)

- Let us determine the spectrum:

  a)

$$\Gamma(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau) e^{-j\omega\tau} = \gamma(0) + \gamma(1) e^{-j\omega} + \gamma(-1) e^{j\omega}$$

$$= \gamma(0) + \gamma(1)\left(e^{-j\omega} + e^{j\omega}\right) = \gamma(0) + 2\gamma(1)\cos\omega$$

$$= \left(1 + c^2\right)\lambda^2 + 2c\lambda^2\cos\omega$$

$$= \left(1 + c^2 + 2c\cos\omega\right)\lambda^2$$

Example 1 (cont.)

- **b)**

$$v(t) = \eta(t) + c\,\eta(t-1) \quad \Longrightarrow \quad W(z) = (1 + cz^{-1})$$

$$\Phi(z) = W(z)W(z^{-1})\lambda^2 = (1 + cz^{-1})(1 + cz)\lambda^2$$

$$= \left[1 + c^2 + c\left(z^{-1} + z\right)\right]\lambda^2$$

But $\Gamma(\omega) = \Phi\left(e^{j\omega}\right)$

$$\Gamma(\omega) = \left[1 + c^2 + c\left(e^{j\omega} + e^{-j\omega}\right)\right]\lambda^2$$

$$= \left(1 + c^2 + 2c\cos\omega\right)\lambda^2$$

Example 1 (cont.)

- **c)**

$$v(t) = \eta(t) + c\,\eta(t-1) \quad \Longrightarrow \quad \begin{aligned} W(z) &= (1 + cz^{-1}) \\ &= \frac{z+c}{z} \end{aligned}$$

But $\Gamma_{vv}(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_{\eta\eta}(\omega)$

$$\Gamma(\omega) = \frac{\left|e^{j\omega} + c\right|^2}{\left|e^{j\omega}\right|^2}\,\lambda^2 = \frac{\left|\cos\omega + j\sin\omega + c\right|^2}{1}\,\lambda^2$$

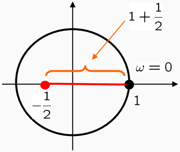
$$= \left[(\cos\omega + c)^2 + (\sin\omega)^2\right]\lambda^2$$

$$= \left[(\cos\omega)^2 + c^2 + 2c\cos\omega \; + (\sin\omega)^2\right]\lambda^2$$

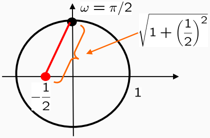
$$= \left(1 + c^2 + 2c\cos\omega\right)\lambda^2$$

Example 1 (cont.)

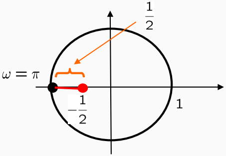
- If $c = \dfrac{1}{2}$



$$\Gamma(0) = \frac{\left(1+\frac{1}{2}\right)^2}{1}\,\lambda^2 = \frac{9}{4}\,\lambda^2$$



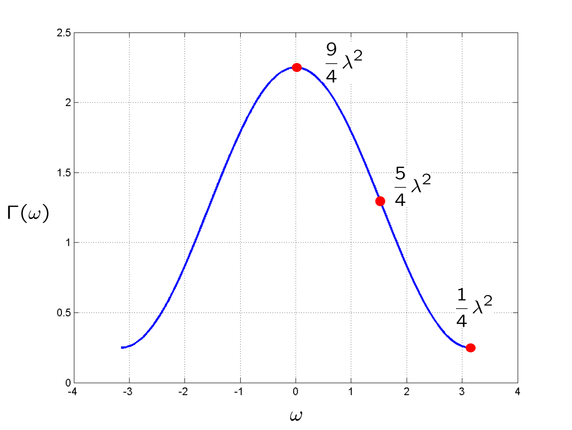
$$\Gamma\left(\frac{\pi}{2}\right) = \frac{1+\left(\frac{1}{2}\right)^2}{1}\,\lambda^2 = \frac{5}{4}\,\lambda^2$$



$$\Gamma(\pi) = \frac{\left(\frac{1}{2}\right)^2}{1}\,\lambda^2 = \frac{1}{4}\,\lambda^2$$

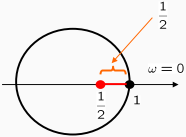
Example 1 (cont.)

- $c = \dfrac{1}{2}$ The “variability” of the process is concentrated at lower frequencies

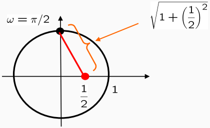


Example 1 (cont.)

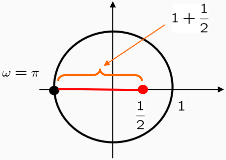
- If $c = -\dfrac{1}{2}$



$$\Gamma(0) = \frac{\left(\frac{1}{2}\right)^2}{1}\,\lambda^2 = \frac{1}{4}\,\lambda^2$$



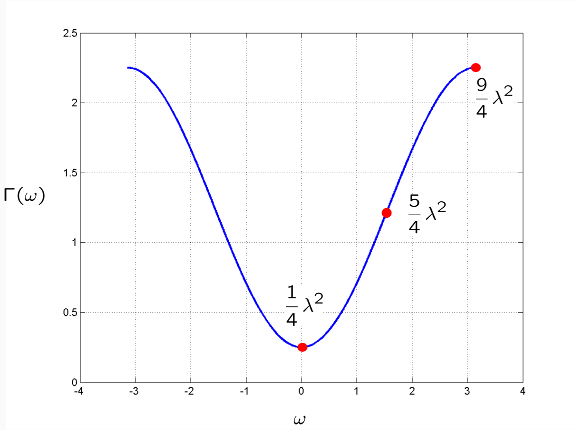
$$\Gamma\left(\frac{\pi}{2}\right) = \frac{1 + \left(\frac{1}{2}\right)^2}{1}\,\lambda^2 = \frac{5}{4}\,\lambda^2$$



$$\Gamma(\pi) = \frac{\left(1 + \frac{1}{2}\right)^2}{1}\,\lambda^2 = \frac{9}{4}\,\lambda^2$$

Example 1 (cont.)

- $c = -\dfrac{1}{2}$ The “variability” of the process is concentrated at higher frequencies



Process MA(∞)

- It is a mathematical abstraction of great conceptual importance that will be used in the sequel
- Consider $\eta(\cdot) \sim WN(0, \lambda^2)$

$$v(t) = \sum_{i=0}^{\infty} c_i \, \eta(t-i) \quad \text{where} \quad c_0 = 1 \qquad (\star)$$

- In order $(\star)$ to represent a well-defined stationary process it is necessary that

$$\operatorname{var}[v(t)] = \left(\sum_{i=0}^{\infty} c_i^2 \right) \lambda^2 < \infty$$

- It is possible to prove that

$$\boxed{\sum_{i=0}^{\infty} c_i^2 < \infty \quad \Longrightarrow \quad \gamma(\tau) < \infty \, , \quad \forall \tau}$$

Analysis of dynamic systems driven by input stationary stochastic processes

Auto-regressive process (AR)

AR process

- Given the white process $\eta(\cdot) \sim \mathsf{WN}(0, \lambda^2)$
- The AR process of order n (denoted with AR(n)) is defined as

$$v(t) = a_1 v(t-1) + a_2 v(t-2) + \cdots + a_n v(t-n) + \eta(t)$$

- Hence $v(t)$ is a linear combination of the values taken on by $v(t)$ itself in the time-window from $t-n$ to $t-1$ plus a white process.

Example: process AR(1)

- Consider the process AR(1):

  $v(t) = a\,v(t-1) + \eta(t)$

$$
\begin{aligned}
v(t) &= a\,v(t-1) + \eta(t) \\
&= a\,\left[a\,v(t-2) + \eta(t-1)\right] + \eta(t) \\
&= a^2\,v(t-2) + a\,\eta(t-1) + \eta(t) \\
&= a^3\,v(t-3) + a^2\eta(t-2) \\
&\qquad\quad + a\,\eta(t-1) + \eta(t) \\
&= \cdots
\end{aligned}
$$

Process AR(1) (cont.)

- Then, in general, for a process AR(1) we can write:

$$v(t) = a\,v(t-1) + \eta(t)$$

$$v(t) = \sum_{i=t_0}^{t-1} a^{t-1-i}\,\eta(i+1) + a^{t-t_0}\,v(t_0) \qquad (\star)$$

- $(\star)$ is consistent with standard linear systems theory according to which the output response can be decomposed in free response (depending only on initial conditions) and forced response (depending only on the input)
- Define

$$\hat{v}(t) = \lim_{t_0 \to -\infty} v(t)$$

- if $|a| < 1$

$$\hat{v}(t) = \sum_{i=-\infty}^{t-1} a^{t-1-i}\,\eta(i+1) = \sum_{j=0}^{\infty} a^{j}\,\eta(t-j)$$

Process AR(1) (cont.)

- But:

$$|a| < 1 \longrightarrow \sum_{j=0}^{\infty} \left(a^j\right)^2 < \infty$$

$$\hat{v}(t) = \sum_{j=0}^{\infty} a^j \, \eta(t-j)$$

Stationary process MA(∞)

The process $\hat{v}(t)$ (stationary of type MA(∞)) is the steady-state solution of the equation of the AR(1) process. Such solution is unique in the context of stationary processes.

Analysis of the process $\hat{v}(t)$

- **Expected value**

$$v(t) = a\,v(t-1) + \eta(t)$$

$$\mathrm{E}[v(t)] = a\;\mathrm{E}[v(t-1)] + \mathrm{E}[\eta(t)]$$

The process is stationary $\bar{v} = a\bar{v}$ $\bar{v} = 0$

- **Variance**

$$\begin{aligned}
\mathrm{E}\left\{[v(t)]^2\right\} &= \mathrm{E}\left\{a^2\,[v(t-1)]^2 + [\eta(t)]^2 + 2a\,v(t-1)\eta(t)\right\} \\
&= \mathrm{E}\left\{a^2\,[v(t-1)]^2\right\} + \mathrm{E}\left\{[\eta(t)]^2\right\} + 2a\,\underbrace{\mathrm{E}\left[v(t-1)\eta(t)\right]}_{=0} \\
&= a^2\,E\left\{[v(t)]^2\right\} + E\left\{[\eta(t)]^2\right\}
\end{aligned}$$

$$(1-a^2)\;\mathrm{E}\left\{[v(t)]^2\right\} = \lambda^2$$

$$\mathrm{E}\left\{[v(t)]^2\right\} = \frac{\lambda^2}{(1-a^2)} \quad (|a|<1)$$

Analysis of the process $\hat{v}(t)$ (cont.)

- **Correlation function**
  - Consider $\tau \geq 0$ ($\gamma(\tau)$ is even).
  - At steady-state the process is MA(∞) and hence we can use the general formula

$$\gamma(\tau) = \lambda^2 \sum_{i=0}^{\infty} c_i c_{i+\tau} = \lambda^2 \sum_{i=0}^{\infty} a^i \, a^{i+\tau} = \lambda^2 \, a^\tau \sum_{i=0}^{\infty} a^{2i}$$

$$= \lambda^2 \, a^\tau \, \frac{1}{(1-a^2)} \quad (|a| < 1)$$

Analysis of the process $\hat{v}(t)$ (cont.)

- An alternative **algebraic** technique to determine the correlation function is:

$$v(t) = a\,v(t-1) + \eta(t)$$

$$v(t)v(t-\tau) = a\,v(t-1)v(t-\tau) + \eta(t)v(t-\tau)$$

$$\mathrm{E}\left[v(t)v(t-\tau)\right] = a\;\mathrm{E}\left[v(t-1)v(t-\tau)\right] + \boxed{\mathrm{E}\left[\eta(t)v(t-\tau)\right]}$$

$$\boxed{\begin{cases} \lambda^2 & \tau = 0 \\ 0 & \tau > 0 \end{cases}}$$

Hence:

- $\tau > 0 \quad \Longrightarrow \quad \gamma(\tau) = a\,\gamma(\tau-1) \quad \Longrightarrow \quad \gamma(\tau) = a^\tau\,\gamma(0)$

$$\gamma(0) = \frac{1}{a}\,\gamma(1)$$

- $\gamma(0) = a\,\gamma(-1) + \lambda^2 = a\,\gamma(1) + \lambda^2$

$$\frac{\gamma(1)}{a} = a\,\gamma(1) + \lambda^2 \quad \Longrightarrow \quad \gamma(1) = \frac{a}{1-a^2}\,\lambda^2$$

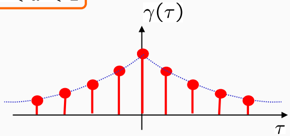
$$\gamma(\tau) = \lambda^2\,a^\tau\,\frac{1}{(1-a^2)} \qquad \tau \geq 0$$

Process AR(1) (cont.)

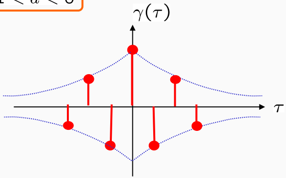
- $R(1): \quad v(t) = a\, v(t-1) + \eta(t)$

$$\gamma(\tau) = \lambda^2\, a^\tau \, \frac{1}{(1-a^2)} \qquad \tau \geq 0$$

$0 < a < 1$



$-1 < a < 0$



- Compared to the case of MA processes, the correlation function vanishes asymptotically (and hence a AR process is “slower” than a MA process).

Process AR(1) (cont.)

- Let us determine the spectrum:

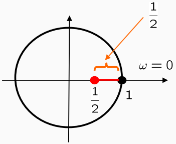
$$v(t) = a\,v(t-1) + \eta(t) \quad \Longrightarrow \quad W(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

But $\Gamma_{vv}(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_{\eta\eta}(\omega)$

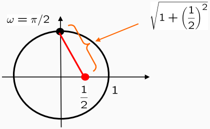
$$\begin{aligned}
\Gamma(\omega) = \frac{\left|e^{j\omega}\right|^2}{\left|e^{j\omega} - a\right|^2}\,\lambda^2 &= \frac{1}{\left|\cos\omega + j\sin\omega - a\right|^2}\,\lambda^2 \\
&= \frac{\lambda^2}{\left[(\cos\omega - a)^2 + (\sin\omega)^2\right]} \\
&= \frac{\lambda^2}{[(\cos\omega)^2 + a^2 - 2a\cos\omega \ + (\sin\omega)^2]} \\
&= \frac{\lambda^2}{(1 + a^2 - 2a\cos\omega)}
\end{aligned}$$

Process AR(1) (cont.)

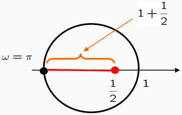
- If $a = \dfrac{1}{2}$



$$\Gamma(0) = \frac{1}{\left(\frac{1}{2}\right)^2}\,\lambda^2 = 4\,\lambda^2$$



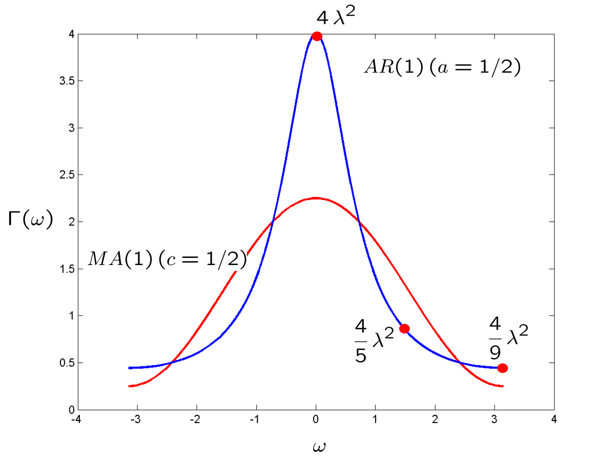
$$\Gamma\left(\frac{\pi}{2}\right) = \frac{1}{1+\left(\frac{1}{2}\right)^2}\,\lambda^2 = \frac{4}{5}\,\lambda^2$$



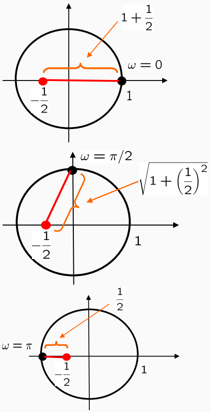
$$\Gamma(\pi) = \frac{1}{\left(1+\frac{1}{2}\right)^2}\,\lambda^2 = \frac{4}{9}\,\lambda^2$$

Process AR(1) (cont.)

- $a = \dfrac{1}{2}$ The "variability" of the process is more concentrated at low-frequencies and the process is "slower" than the "analogous" MA



- If $a = -\dfrac{1}{2}$



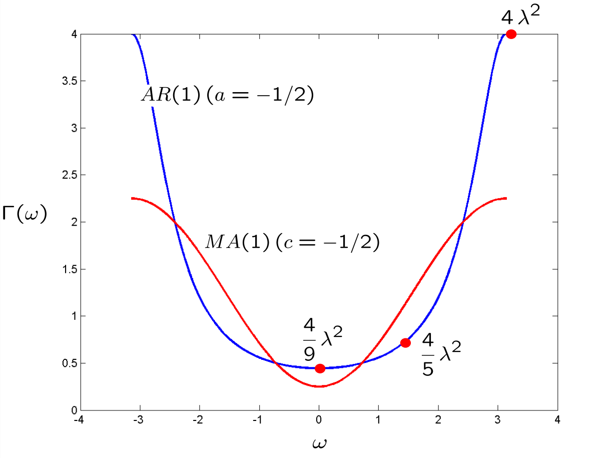
$$\Gamma(0) = \frac{1}{\left(1+\frac{1}{2}\right)^2}\,\lambda^2 = \frac{4}{9}\,\lambda^2$$

$$\Gamma\left(\frac{\pi}{2}\right) = \frac{1}{1+\left(\frac{1}{2}\right)^2}\,\lambda^2 = \frac{4}{5}\,\lambda^2$$

$$\Gamma(\pi) = \frac{1}{\left(\frac{1}{2}\right)^2}\,\lambda^2 = 4\,\lambda^2$$

Process AR(1) (cont.)

- $a = -\frac{1}{2}$ The "variability" of the process is more concentrated at high frequency and the frequency distribution is very different from the "analogous" MA



Example: process AR(2)

- Consider the process AR(2)

$$v(t) = a_1\, v(t-1) + a_2\, v(t-2) + \eta(t)$$

$$\begin{aligned} v(t)v(t-\tau) = & a_1\, v(t-1)v(t-\tau) \\ & + a_2\, v(t-2)v(t-\tau) + \eta(t)v(t-\tau) \end{aligned}$$

Hence:

$$\begin{aligned} \gamma(\tau) &= \mathrm{E}\left[v(t)v(t-\tau)\right] \\ &= a_1\ \mathrm{E}\left[v(t-1)v(t-\tau)\right] \\ &+ a_2\ \mathrm{E}\left[v(t-2)v(t-\tau)\right] + \boxed{\mathrm{E}\left[\eta(t)v(t-\tau)\right]} \end{aligned}$$

$$\begin{cases} \lambda^2 & \tau = 0 \\ 0 & \tau > 0 \end{cases}$$

- Then
  - $\tau > 0 \quad \Longrightarrow \quad \gamma(\tau) = a_1\, \gamma(\tau-1) + a_2\, \gamma(\tau-2)$
  - $\gamma(0) = a_1\, \gamma(-1) + a_2\, \gamma(-2) + \lambda^2 = a_1\, \gamma(1) + a_2\, \gamma(2) + \lambda^2$

Yule-Walker equations

- The useful equations are:

$$\gamma(2) = a_1\,\gamma(1) + a_2\,\gamma(0)$$
$$\gamma(1) = a_1\,\gamma(0) + a_2\,\gamma(1)$$
$$\gamma(0) = a_1\,\gamma(1) + a_2\,\gamma(2) + \lambda^2$$

- These equations can be organized in matrix form:

$$\begin{bmatrix} a_2 & a_1 & -1 \\ a_1 & a_2 - 1 & 0 \\ 1 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda^2 \end{bmatrix} \qquad (\star)$$

- Then, for given $a_1\,,\ a_2\,,\ \lambda^2$ it is possible to compute $\gamma(0)\ \ \gamma(1)\,,\ \gamma(2)$ and afterwards proceed in a recursive way.
- Equations $(\star)$ are the well-known **Yule-Walker equations** and can be written for any generic AR process.

Analysis in case of non-zero mean white noise

- Consider the process

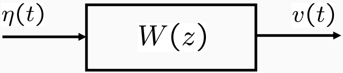
  $v(t) = a\, v(t-1) + \eta(t)$ with $\eta(\cdot) \sim WN(\bar{\eta}, \lambda^2)$

$$
\begin{aligned}
\mathrm{E}[v(t)] &= a\ \mathrm{E}[v(t-1)] + \mathrm{E}[\eta(t)] \\
&= a\ \mathrm{E}[v(t-1)] + \bar{\eta}
\end{aligned}
$$

- Recalling that $|a| < 1$ and setting $\mathrm{E}\left[v(t)\right] = \bar{v}$

$$\bar{v} = a\, \bar{v} + \bar{\eta} \quad \Longrightarrow \quad \bar{v} = \frac{\bar{\eta}}{1-a}$$

- Notice that



$$W(z) = \frac{z}{z-a}$$

$$\bar{v} = W(1)\ \bar{\eta}$$

Static gain

Analysis in case of non-zero mean white noise (cont.)

- Let us determine the covariance function:

$$\gamma(\tau) = E\left\{[v(t) - \bar{v}][v(t-\tau) - \bar{v}]\right\}$$

- Introduce the process

$$\tilde{v}(t) = v(t) - \bar{v} \quad \Longrightarrow \quad v(t) = \tilde{v}(t) + \bar{v}$$

and hence

$$\begin{aligned}\gamma(\tau) &= \mathrm{E}\left\{[v(t) - \bar{v}][v(t-\tau) - \bar{v}]\right\} \\ &= \mathrm{E}\left\{[\tilde{v}(t) + \cancel{\bar{v}} - \cancel{\bar{v}}][\tilde{v}(t-\tau) + \cancel{\bar{v}} - \cancel{\bar{v}}]\right\}\end{aligned}$$

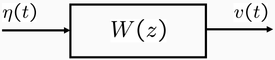
> The correlation function of the zero-mean process $\tilde{v}(t)$ coincides with the covariance of the original process.

In general:

- Consider the process

$$AR(n): \quad v(t) = a_1\, v(t-1) + \cdots + a_n v(t-n) + \eta(t)$$

$$A(z)v(t) = \eta(t) \text{ with } A(z) = 1 - a_1 z^{-1} - \cdots - a_n z^{-n}$$



$$W(z) = \frac{1}{A(z)} = \frac{1}{1 - a_1 z^{-1} - \cdots - a_n z^{-n}} = \frac{z^n}{z^n - a_1 z^{n-1} - \cdots - a_n}$$

- If the roots of $A(z)$ (i.e., the poles of $W(z)$) are all strictly located inside the unit-circle, then in steady-state we obtain a stationary process equivalent to a process MA(∞) .

Analysis of dynamic systems driven by input stationary stochastic processes

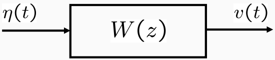
Auto-regressive and Moving average process (ARMA)

ARMA processes

- Given a white process $\eta(\cdot) \sim WN(0, \lambda^2)$
- An ARMA
  process of order $n_a\,,\ n_c$ (and denoted ARMA$(n_a\,,\ n_c)$) is given by

$$
\begin{aligned}
v(t) = & a_1 v(t-1) + a_2 v(t-2) + \cdots + a_n v(t-n) \\
& + \eta(t) + c_1 \eta(t-1) + c_2 \eta(t-2) + \cdots + c_n \eta(t-n)
\end{aligned}
$$

$$
A(z)v(t) = C(z)\eta(t) \quad \text{with} \quad
\begin{aligned}
A(z) = & 1 - a_1 z^{-1} - \cdots - a_{n_a} z^{-n_a} \\
C(z) = & 1 + c_1 z^{-1} + \cdots + c_{n_c} z^{-n_c}
\end{aligned}
$$



$$
W(z) = \frac{C(z)}{A(z)} = \frac{1 + c_1 z^{-1} + \cdots + c_{n_c} z^{-n_c}}{1 - a_1 z^{-1} - \cdots - a_{n_a} z^{-n_a}}
$$

If $n = \max(n_a, n_c)$ **maximum input/output delay**

$$
W(z) = \frac{z^n + c_1 z^{n-1} + \cdots + c_{n_c} z^{n-n_c}}{z^n - a_1 z^{n-1} - \cdots - a_{n_a} z^{n-n_a}}
$$

ARMA processes (cont.)

- If the stability condition is satisfied (all roots of $A(z)$ are strictly inside the unity circle) then in steady-state we obtain a stationary process equivalent to a process MA(∞)

$$W(z) = \frac{C(z)}{A(z)} = w_0 + w_1 z^{-1} + \cdots + w_i z^{-i} + \cdots$$

where $h(t) = w_t\,, \quad t = 0, 1, \ldots$ is the impulse response of the dynamic system

$$\text{imp}(t) \longrightarrow \boxed{W(z)} \longrightarrow h(t)$$

- If the stability condition is satisfied $\Longrightarrow \displaystyle\sum_{j=0}^{\infty} (w_j)^2 < \infty$

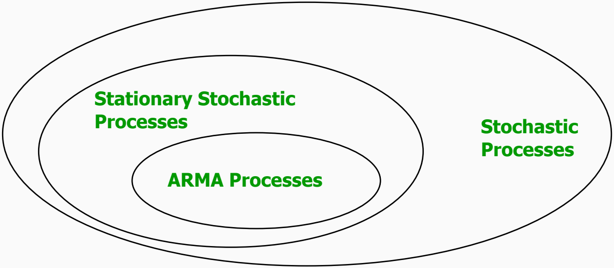
$\Longrightarrow$ process variance of MA(∞) is finite

Remarks

- With reference to a generic stationary stochastic process:

$$\text{Difference-equation} \Longleftrightarrow \text{Process model}$$
$$\text{Stationary solution of difference-equation} \Longleftrightarrow \text{Process}$$

- In general:



Example

- Consider the process $ARMA(1,1)$

$$v(t) = \frac{1}{2}\,v(t-1) + \eta(t) + \frac{1}{3}\eta(t-1)\,, \quad \eta(\cdot) \sim WN(0,1)$$

$$\left(1 - \frac{1}{2}z^{-1}\right) v(t) = \left(1 + \frac{1}{3}z^{-1}\right)\eta(t)$$

$$v(t) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}\,\eta(t) = \frac{z + \frac{1}{3}}{z - \frac{1}{2}}\,\eta(t)$$

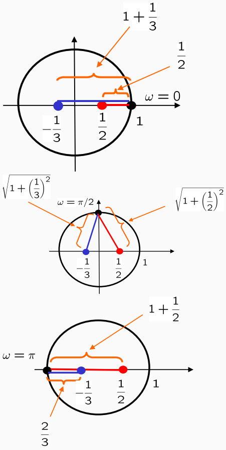
- Let us determine the spectrum:

$$\Gamma(\omega) = \left|W(e^{j\omega})\right|^2 \lambda^2 = \frac{\left|e^{j\omega} + \frac{1}{3}\right|^2}{\left|e^{j\omega} - \frac{1}{2}\right|^2} = \frac{\left|\cos\omega + j\sin\omega + \frac{1}{3}\right|^2}{\left|\cos\omega + j\sin\omega - \frac{1}{2}\right|^2}$$

$$= \frac{(\cos\omega)^2 + \frac{1}{9} + \frac{2}{3}\cos\omega \; + (\sin\omega)^2}{(\cos\omega)^2 + \frac{1}{4} - \cos\omega \; + (\sin\omega)^2}$$

$$= \frac{\frac{10}{9} + \frac{2}{3}\cos\omega}{\frac{5}{4} - \cos\omega}$$

Example (cont.)

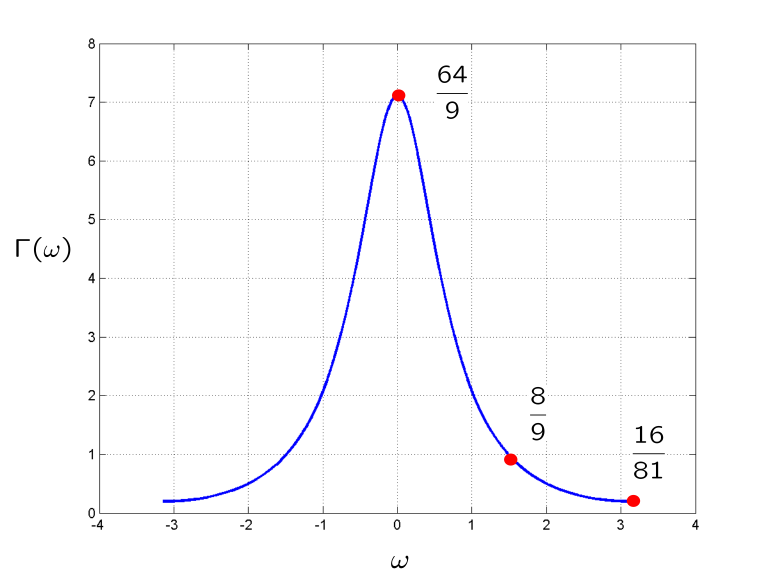


$$\Gamma(0) = \frac{\left(1+\frac{1}{3}\right)^2}{\left(\frac{1}{2}\right)^2} = \frac{64}{9}$$

$$\Gamma\left(\frac{\pi}{2}\right) = \frac{1+\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{2}\right)^2} = \frac{8}{9}$$

$$\Gamma(\pi) = \frac{\left(\frac{2}{3}\right)^2}{\left(1+\frac{1}{2}\right)^2} = \frac{16}{81}$$

Example (cont.)

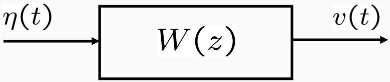


Analysis of dynamic systems driven by input stationary stochastic processes

Spectral factorization

Spectral factorization

- A few families of stationary stochastic process with rational spectra have been described. The prediction problem will be addressed in the context of these families.
- However: some further discussions are necessary on the representation of st. processes with rational spectra.
- Consider



with $\eta(\cdot) \sim \text{WN}(0, \lambda^2)$ and $W(z) = \dfrac{N(z)}{D(z)}$ rational

Spectral factorization (cont.)

Fundamental question:

Does $\tilde{W}(z)$ exist such that, for a suitable input process, $\tilde{\eta}(\cdot) \sim \text{WN}(0, \tilde{\lambda}^2)$ we have

$$\gamma_{vv}(\tau) = \gamma_{\tilde{v}\tilde{v}}(\tau)\,,\ \forall\,\tau \qquad \text{that is} \qquad \Gamma(\omega) = \tilde{\Gamma}(\omega)\,,\ \forall\,\omega$$

In qualitative terms, does another transfer function exist yielding the same correlation function/spectrum?

- The question is important: in case of existence of such other transfer function $\tilde{W}(z)$, this would imply the existence of *more than one* rational representation of the same stochastic process
- Trying to estimate a transfer function (for example, a predictor) on the basis of experimental data would be a ill-posed problem

Spectral factorization (cont.)

- Then, let us try to understand the mutual relation between transfer functions of equivalent representations of the same process.
- More details are needed on the representation of processes with rational spectra.
- Recall that $\Phi(z) = W(z)W\left(z^{-1}\right)\lambda^2$ where $\Phi(z)$ is the complex spectrum

Spectral factorization problem

Given a complex spectrum $\Phi(z)$, determine all pairs $\left[W(z), \lambda^2\right]$ such that $W(z)W\left(z^{-1}\right)\lambda^2 = \Phi(z)$

Spectral factorization (cont.)

- Let us now analyze the ways to modify $W(z)$ without modifying $\Phi(z)$

  (a) $\alpha \cdot W(z)$ with $\alpha \neq 0 \qquad \Longrightarrow \qquad \Phi(z) = \alpha W(z) \alpha W(z^{-1}) \tilde{\lambda}^2$

> Choosing $\tilde{\lambda}^2 = \dfrac{\lambda^2}{\alpha^2}$ we have that the pairs
>
> $\left[W(z), \lambda^2\right]$ and $\left[\alpha W(z), \dfrac{\lambda^2}{\alpha^2}\right]$ with $\alpha \neq 0$
>
> have the same $\Phi(z)$

This result is not surprising: the variance of a stationary process can be changed either by acting on the static gain of the transfer function and on the variance of the input process.

Spectral factorization (cont.)

(b) $z^{-k}\, W(z)$ with $k \neq 0$

$$\Phi(z) = z^{-k}W(z)z^{k}W(z^{-1})\lambda^2 = W(z)W(z^{-1})\lambda^2$$

The pairs $\left[W(z), \lambda^2\right]$ and $\left[z^{-k}\, W(z), \lambda^2\right]$ have the same $\Phi(z)$

Also this result is not surprising: multiplying by z^{-k} means considering realizations delayed by time-steps and this clearly does not alter the probabilistic features of the stochastic process.

Spectral factorization (cont.)

(c) trivial case

$$\frac{(z+a)^n}{(z+a)^n} W(z) \quad \text{with} \quad a \in \mathbb{C},\, n \geq 1$$

The pairs $\left[W(z), \lambda^2\right]$ and $\left[\frac{(z+a)^n}{(z+a)^n} W(z), \lambda^2\right]$

have the same $\Phi(z)$

Spectral factorization (cont.)

(c) non-trivial case $\dfrac{1}{a}\,\dfrac{z+a}{z+\frac{1}{a}}\,W(z)$ with $a \in \mathbb{C}$

$$T(z)T(z^{-1}) = \varrho\,\frac{z+a}{z+\frac{1}{a}}\,\varrho\,\frac{z^{-1}+a}{z^{-1}+\frac{1}{a}} = \varrho^2\,\frac{(z+a)(z^{-1}+a)}{(z+\frac{1}{a})(z^{-1}+\frac{1}{a})}$$

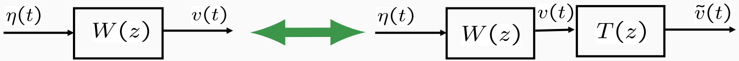
$$= \varrho^2\,\frac{1+a^2+a(z+z^{-1})}{1+\frac{1}{a^2}+\frac{1}{a}(z+z^{-1})} = \varrho^2\,a^2\,\frac{1+a^2+a(z+z^{-1})}{1+a^2+a(z+z^{-1})} = \varrho^2\,a^2$$

Choosing $\varrho = \dfrac{1}{a}$ the pairs $\left[W(z), \lambda^2\right]$ and

$\left[\dfrac{1}{a}\,\dfrac{z+a}{z+\frac{1}{a}}\,W(z), \lambda^2\right]$ have the same $\Phi(z)$

An "all-pass" filter

- Then, if $T(z) = \dfrac{1}{a} \dfrac{z+a}{z+\dfrac{1}{a}}$



in the sense that the spectra of $v(t)$ and of $\tilde{v}(t)$ coincide.

Therefore, canceling a pole (zero) with a reciprocal zero (pole) leaves the spectrum unchanged (except for a possible multiplicative constant)

Spectral factorization (cont.)

- The problem we are addressing is the one of guaranteeing the uniqueness of the representation of the stationary stochastic process that is, that there exists a **unique** transfer function $W(z)$ for which the process can be represented as the output of a linear system with transfer function $W(z)$ and a white process as input.
- Clearly, there are many ways to constrain the representation to be unique. The ways we are considering are the ones that will be useful in the context of the solution of the prediction problem.

Spectral factorization (cont.)

- With reference to cases (a), (b), (c) previously addressed:
  - given $W(z) = \frac{N(z)}{D(z)}$
  - (a): it is sufficient to set some parameter. We impose that $N(z)$ and $D(z)$ are monic polynomials
  - (b): we impose that $N(z)$ and $D(z)$ have the degree a-priori set (*for example*, the same degree)
  - (c): we impose that $N(z)$ and $D(z)$ are co-prime and that all zeros and poles are inside the unit circle

Spectral factorization theorem

Spectral factorization theorem

Given a process with rational spectrum $\Phi(z)$, there exists one and only one representation of the process as the output of a linear system driven by a white process and with transfer function $W(z) = \dfrac{N(z)}{D(z)}$ if the following conditions are imposed on $W(z)$:

- $N(z)$ and $D(z)$ monic, co-prime and of the same degree
- all roots of $N(z)$ (zeros of $W(z)$) have $|\cdot| \leq 1$
- all roots of $D(z)$ (poles of $W(z)$) have $|\cdot| < 1$

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Lecture 7

Dynamic models of stationary discrete-time stochastic processes

END