More Definitions and Properties

- **Time-invariant Dynamic Systems**

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), \cancel{t}) \\ y(t) = g(x(t), u(t), \cancel{t}) \end{cases} \Longrightarrow \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k), u(k), \cancel{k}) \\ y(k) = g(x(k), u(k), \cancel{k}) \end{cases} \Longrightarrow \begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = g(x(k), u(k)) \end{cases}$$

- **Strictly Proper Dynamic Systems**

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), \cancel{u(t)}, t) \end{cases} \Longrightarrow \begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), t) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), \cancel{u(k)}, k) \end{cases} \Longrightarrow \begin{cases} x(k+1) = f(x(k), u(k), k) \\ y(k) = g(x(k), k) \end{cases}$$

More Definitions and Properties (cont.)

- **Forced and Free Dynamic Systems**

$$\begin{cases} \dot{x}(t) = f(x(t),\ \cancel{u(t)}\ ,t) \\ y(t) = g(x(t),\ \cancel{u(t)}\ ,t) \end{cases} \Longrightarrow \begin{cases} \dot{x}(t) = f(x(t),t) \\ y(t) = g(x(t),t) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k),\ \cancel{u(k)}\ ,k) \\ y(k) = g(x(k),\ \cancel{u(k)}\ ,k) \end{cases} \Longrightarrow \begin{cases} x(k+1) = f(x(k),k) \\ y(k) = g(x(k),k) \end{cases}$$

It is worth noting that in case the input function $u(t),\ \forall\, t$ or input sequence $u(k),\ \forall\, k$ are **known beforehand**, the dynamic system can be re-written as a free one:

$$\begin{cases} \dot{x}(t) = f(x(t),u(t),t) = \widetilde{f}(x(t),t) \\ y(t) = g(x(t),u(t),t) = \widetilde{g}(x(t),t) \end{cases}$$

$$\begin{cases} x(k+1) = f(x(k),u(k),k) = \widetilde{f}(x(k),k) \\ y(k) = g(x(k),u(k),k) = \widetilde{g}(x(k),k) \end{cases}$$

$u(k)$

fissa

More Definitions and Properties (cont.)

definizioni

- **Free Movement**

$$\begin{aligned}
&\dot{x}(t) = f(x(t), u(t), t) \\
&y(t) = g(x(t), u(t), t) \\
&\text{with:} \\
&\quad x(t_0) = x_0\,;\; u(t) = 0\,,\;\; \forall\, t
\end{aligned}
\quad \Longrightarrow \quad
\begin{gathered}
\{\, (x_l(t), t),\; t \in [t_0, t_1]\,\} \\
\text{free movement}
\end{gathered}$$

$$\begin{aligned}
&x(k+1) = f(x(k), u(k), k) \\
&y(k) = g(x(k), u(k), k) \\
&\text{with:} \\
&\quad x(k_0) = x_0\,;\; u(k) = 0\,,\;\; \forall\, k
\end{aligned}
\quad \Longrightarrow \quad
\begin{gathered}
\{\, (x_l(k), k),\; k \in [k_0, k_1]\,\} \\
\text{free movement}
\end{gathered}$$

More Definitions and Properties (cont.)

definizione

- **Forced Movement**

$$\dot{x}(t) = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$
with:
$$x(t_0) = 0$$

$$\Longrightarrow \quad \{\,(x_f(t), t),\; t \in [t_0, t_1]\,\}$$
forced movement

$$x(k+1) = f(x(k), u(k), k)$$
$$y(k) = g(x(k), u(k), k)$$
with:
$$x(k_0) = 0$$

$$\Longrightarrow \quad \{\,(x_f(k), k),\; k \in [k_0, k_1]\,\}$$
forced movement

Discrete-time Systems

Consider:

$k \in \mathbb{Z}$

$$\begin{aligned} x(k+1) &= f(x(k), u(k), k) \\ y(k) &= g(x(k), u(k), k) \end{aligned} \quad , \qquad k > k_0,\ x(k_0) = x_0$$

Clearly, by iterating the state equations:

$$\begin{aligned}
x(k_0) &= x_0 \\
x(k_0+1) &= f(x(k_0), u(k_0), k_0) \\
x(k_0+2) &= f(x(k_0+1), u(k_0+1), k_0+1) \\
&= f(f(x(k_0), u(k_0), k_0), u(k_0+1), k_0+1) \\
x(k_0+3) &= f(x(k_0+2), u(k_0+2), k_0+2) \\
&= f(f(f(x(k_0), u(k_0), k_0), u(k_0+1), k_0+1), u(k_0+2), k_0+2)
\end{aligned}$$

and so on. Hence, the **state transition function** has the form

$$x(k) = \varphi(k, k_0, x_0, \{u(k_0), \ldots, u(k-1)\})$$

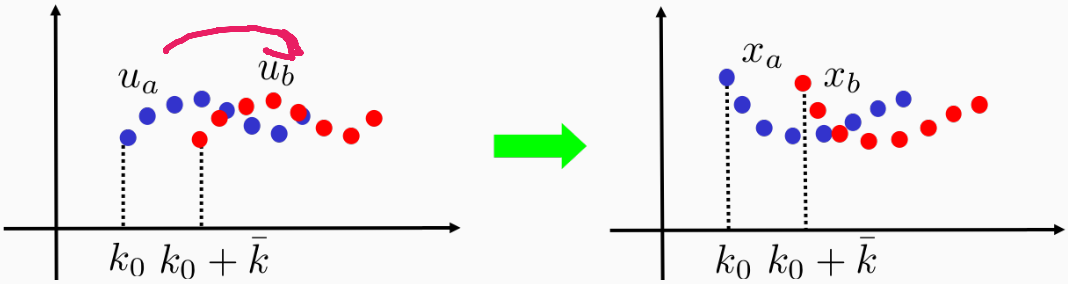
thus enhancing the **causality property**.

Time-invariant Discrete-time Systems

$$\begin{aligned} x(k+1) &= f(x(k), u(k), \cancel{k}) \\ y(k) &= g(x(k), u(k), \cancel{k}) \end{aligned} \quad , \; \underline{x(k_0) = x_0,\; u_a(k) = u(k),\; k \in \{k_0, \dots, k_1\}}$$

yields the state sequence $x_a(k),\; k \in \{k_0, \dots, k_1\}$. Let's shift the initial time by $\bar{k}$ and the input sequence as well:

$$\begin{aligned} & x(k_0 + \bar{k}) = x_0 \\ & u_b(k) = u_a(k - \bar{k}), \\ & \quad k \in \{k_0 + \bar{k}, \dots, k_1 + \bar{k}\} \end{aligned} \implies \begin{aligned} & x_b(k) = x_a(k - \bar{k}), \\ & \quad k \in \{k_0 + \bar{k}, \dots, k_1 + \bar{k}\} \end{aligned}$$



Conventionally, we set $k_0 = 0$.

Equilibrium Analysis: Equilibrium States and Outputs

- A state $\bar{x} \in \mathbb{R}^n$ is an **equilibrium state** if $\forall\, k_0$, $\exists\, \{\bar{u}(k) \in \mathbb{R}^m, k \geq k_0\}$ such that

$$\begin{array}{l} x(k_0) = \bar{x} \\ u(k) = \bar{u}(k),\ \forall\, k \geq k_0 \end{array} \implies \boxed{x(k) = \bar{x},\ \forall\, k > k_0}$$

- An output $\bar{y} \in \mathbb{R}^p$ is an **equilibrium output** if $\forall\, k_0$, $\exists\, \{\bar{u}(k) \in \mathbb{R}^m, k \geq k_0\}$ such that

$$\begin{array}{l} x(k_0) = \bar{x} \\ u(k) = \bar{u}(k),\ \forall\, k \geq k_0 \end{array} \implies y(k) = \bar{y},\ \forall\, k > k_0$$

In general:

- The input sequence $\{\bar{u}(k) \in \mathbb{R}^m, k \geq k_0\}$ depends on the initial time k_0
- The fact that the state is of equilibrium does **not** imply that the corresponding output coincides with an equilibrium output

Equilibrium Analysis in the Time-invariant Case

In the time-invariant case, **all equilibrium states** can be determined by imposing **constant** input sequences.

A state $\bar{x} \in \mathbb{R}^n$ is an equilibrium state if $\exists\, \bar{u} \in \mathbb{R}^m$ such that

$$\begin{array}{l} x(k_0) = \bar{x} \\ u(k) = \bar{u},\ \forall\, k \geq k_0 \end{array} \quad \Longrightarrow \quad x(k) = \bar{x},\ \forall\, k > k_0$$

All equilibrium states $\bar{x} \in \mathbb{R}^n$ can thus be obtained by finding all solutions of the algebraic equation

$$\bar{x} = f(\bar{x}, \bar{u})\,, \quad \forall\, \bar{u} \in \mathbb{R}^m$$

The following sets are also introduced:

$$\begin{aligned} \bar{X}_{\bar{u}} &= \{\bar{x} \in \mathbb{R}^n :\ \bar{x} = f(\bar{x}, \bar{u})\} \\ \bar{X} &= \{\bar{x} \in \mathbb{R}^n :\ \exists\, \bar{u} \in \mathbb{R}^m \text{ such that } \bar{x} = f(\bar{x}, \bar{u})\} \end{aligned}$$

State Space Descriptions

But ... How to determine a state space description?

Recall:

State variables

Variables to be known at time $t = t_0$ in order to be able to determine the output $y(t),\ t \geq t_0$ from the knowledge of the input $u(t),\ t \geq t_0$:

$$x_i(t),\ i = 1, 2, \ldots, n \qquad \text{(state variables)}$$

State Space Descriptions(cont.)

A "physical" criterion

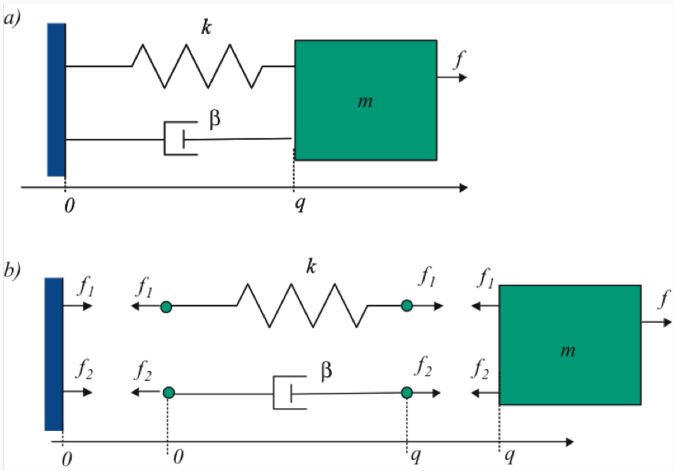
State variables can be defined as entities associated with storage of mass, energy, etc. . . .

For example:

- **Passive electrical systems**: voltages on capacitors, currents on inductors
- **Translational mechanical systems**: linear displacements and velocities of each independent mass
- **Rotational mechanical systems**: angular displacements and velocities of each independent inertial rotating mass
- **Hydraulic systems**: pressure or level of fluids in tanks
- **Thermal systems**: temperatures
- . . .

State Space Descriptions: Example 1 (continuous-time)

A mechanical system

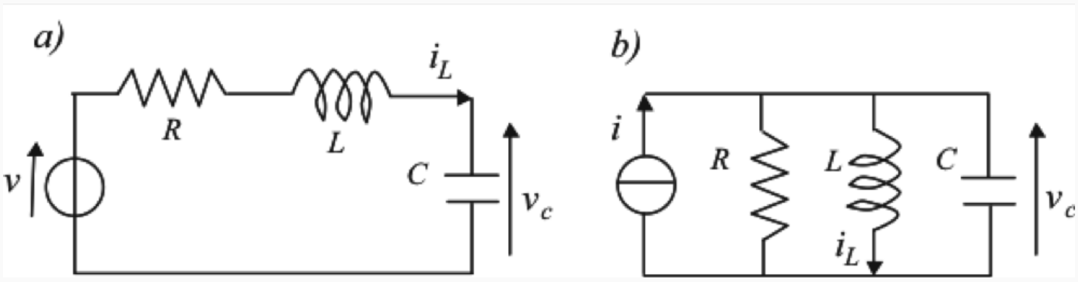


$$m\ddot{q} + \beta\dot{q} + kq = f$$

$$\begin{aligned} x_1 &:= q \\ x_2 &:= \dot{q} \end{aligned} \quad \Longrightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{q} = -\dfrac{k}{m}x_1 - \dfrac{\beta}{m}x_2 + \dfrac{1}{m}f \end{cases}$$

State Space Descriptions: Example 2 (continuous-time)

Electrical systems



$$L\frac{di_L}{dt} = v - Ri_L - v_C \qquad\qquad C\frac{dv_C}{dt} = i - \frac{1}{R}v_C - i_L$$

$$C\frac{dv_C}{dt} = i_L \qquad\qquad L\frac{di_L}{dt} = v_C$$

$$x_1 := i_L\,;\;\; x_2 := v_C$$

$$\begin{cases} \dot{x}_1 = -\dfrac{R}{L}x_1 - \dfrac{1}{L}x_2 + \dfrac{1}{L}v \\ \dot{x}_2 = \dfrac{1}{C}x_1 \end{cases} \qquad\qquad \begin{cases} \dot{x}_1 = \dfrac{1}{L}x_2 \\ \dot{x}_2 = -\dfrac{1}{C}x_1 - \dfrac{1}{RC}x_2 + \dfrac{1}{C}iv \end{cases}$$

State Space Descriptions: Example 3 (discrete-time)

Student dynamics: 3-years undergraduate course

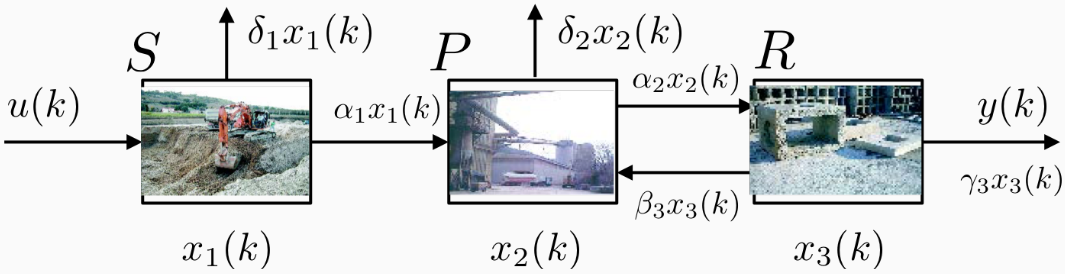
- percentages of students promoted, repeaters, and dropouts are roughly constant
- direct enrolment in 2nd and 3rd academic year is not allowed
- students cannot enrol for more than 3 years

$$\begin{cases} x_1(k+1) = \beta_1 x_1(k) + u(k) \\ x_2(k+1) = \alpha_1 x_1(k) + \beta_2 x_2(k) \\ x_3(k+1) = \alpha_2 x_2(k) + \beta_3 x_3(k) \\ y(k) = \alpha_3 x_3(k) \end{cases}$$

- $x_i(k)$: number of students enrolled in year i at year k, $i = 1, 2, 3$
- $u(k)$: number of freshmen at year k
- $y(k)$: number of graduates at year k
- α_i: promotion rate during year i, $\alpha_i \in [0, 1]$
- β_i: failure rate during year i, $\beta_i \in [0, 1]$
- γ_i: dropout rate during year i, $\gamma_i = 1 - \alpha_i - \beta_i \geq 0$

State Space Descriptions: Example 4 (discrete-time)

Supply chain



- S purchases the quantity $u(k)$ of raw material at each month k
- A fraction δ_1 of raw material is discarded, a fraction α_1 is shipped to producer P
- A fraction α_2 of product is sold by P to retailer R, a fraction δ_2 is discarded
- Retailer R returns a fraction β_3 of defective products every month, and sells a fraction γ_3 to customers

State Space Descriptions: Example 4 (discrete-time) (cont.)

$$\begin{cases} x_1(k+1) = (1-\alpha_1-\delta_1)x_1(k) + u(k) \\ x_2(k+1) = \alpha_1 x_1(k) + (1-\alpha_2-\delta_2)x_2(k) \\ \qquad\qquad\qquad +\beta_3 x_3(k) \\ x_3(k+1) = \alpha_2 x_2(k) + (1-\beta_3-\gamma_3)x_3(k) \\ y(k) = \gamma_3 x_3(k) \end{cases}$$

- k: month counter
- $x_1(k)$: raw material in S
- $x_2(k)$: products in P
- $x_3(k)$: products in R
- $y(k)$: products sold to customers

State Space Descriptions (cont.)

A "mathematical" criterion

- **Continuous-time case**. An input-out differential equation model of the system is available:

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = \varphi\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}, \ldots, \frac{\mathrm{d}y}{\mathrm{d}t}, y, u, t\right)$$

- **Discrete-time case**. An input-out difference equation model of the system is available:

$$y(k+n) = \varphi\left(y(k+n-1), y(k+n-2), \ldots, y(k), u(k), k\right)$$

Suitable state variables – without necessarily a physical meaning – are **defined** to represent "mathematically" the differential equation or the difference equation models of the dynamic system

State Space Descriptions (cont.)

Continuous-time case:

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = \varphi\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}, \ldots, \frac{\mathrm{d}y}{\mathrm{d}t}, y, u, t\right)$$

Letting:

$$\begin{cases} x_1(t) := y(t) \\ x_2(t) := \dfrac{\mathrm{d}y}{\mathrm{d}t} \\ \vdots \\ x_n(t) := \dfrac{\mathrm{d}^n y}{\mathrm{d}t^n} \end{cases} \implies \quad x := \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

one gets:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_n = \varphi(x, u, t) \\ y = x_1 \end{cases}$$