Systems Dynamics

Course ID: 267MI – Fall 2018

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Lecture 10
Solution of the Prediction Problem

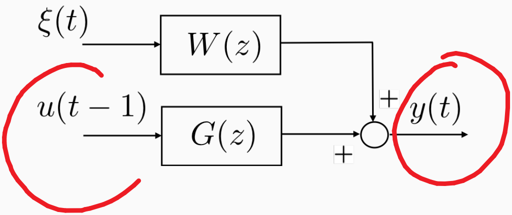
Models and Predictors

Models and Predictors

- Consider the general model

$$\mathcal{M}(\vartheta): \quad y(t) = G(z)\, u(t-1) + W(z)\, \xi(t)$$

where ϑ denotes a **vector of parameters characterizing the model** in which the one-step delay between input and output is explicitly enhanced (a widely used convention)



$$A(z)\,y(t) = B(z)\,u(t-1) + \varepsilon(t) \qquad \varepsilon \sim WN$$

$$A(z) = 1 - a_1 z^{-1} - \dots a_N z^{-N}$$

$$B(z) = b_0 + b_1 z^{-1} + \dots$$

$$G(z)\,u(t-1) \Leftrightarrow z^{-1} G(z)\,u(t)$$

$$\frac{N_G(z)}{z D_G(z)}\,u(t)$$

Models and Predictors (cont.)

- Let us determine the optimal predictor:

$W(z)$ $|\text{poli}|<1$ $|\text{zeri}|<1$

$$
\begin{aligned}
y(t) &= G(z)\,u(t-1) + W(z)\,\xi(t) \\
\Longrightarrow\ \frac{1}{W(z)}\,y(t) &= \frac{G(z)}{W(z)}\,u(t-1) + \xi(t) \\
\Longrightarrow\ y(t) + \frac{1}{W(z)}\,y(t) &= y(t) + \frac{G(z)}{W(z)}\,u(t-1) + \xi(t) \\
\Longrightarrow\ y(t) &= \left[1 - \frac{1}{W(z)}\right] y(t) + \frac{G(z)}{W(z)}\,u(t-1) + \xi(t)
\end{aligned}
$$

- But $W(z)$ is monic and hence $1 - \dfrac{1}{W(z)} = \# z^{-1} + \# z^{-2} + \cdots$.
- Therefore, $\left[1 - \dfrac{1}{W(z)}\right] y(t)$ depends on $y(t-1), y(t-2), \ldots$.
- Moreover, $\dfrac{G(z)}{W(z)}\,u(t-1)$ depends on $u(t-1), u(t-2), \ldots$

Models and Predictors (cont.)

- Therefore, since $\xi(t)$ is white, the class of optimal predictors $\widehat{\mathcal{M}}(\vartheta)$ associated with the class of models $\mathcal{M}(\vartheta)$ is:

$$\widehat{\mathcal{M}}(\vartheta): \quad \hat{y}(t \,|\, t-1) = \left[1 - \frac{1}{W(z)}\right] y(t) + \frac{G(z)}{W(z)}\, u(t-1)$$

  where the optimality stems from the fact that the prediction error

$$\boxed{\hat{\varepsilon}(t) = y(t) - \hat{y}(t \,|\, t-1) = \xi(t)}$$

  is white (zero expected value and variance equal to the variance of $\xi(t)$).

- Let us now consider another predictor $\widetilde{\mathcal{M}}(\vartheta)$ with a white prediction error $\tilde{\varepsilon}(t)$ with zero expected value. Assume that $\widetilde{\mathcal{M}}(\vartheta)$ is “better” than $\widehat{\mathcal{M}}(\vartheta)$, that is

$$\mathrm{var}\left[\tilde{\varepsilon}(t)\right] < \mathrm{var}\left[\hat{\varepsilon}(t)\right]$$

Models and Predictors (cont.)

- But:

$$\begin{aligned}\tilde{\varepsilon}(t) &= y(t) - \tilde{y}(t\,|\,t-1) = y(t) - \hat{y}(t\,|\,t-1) + \hat{y}(t\,|\,t-1) - \tilde{y}(t\,|\,t-1) \\ &= \xi(t) + \hat{y}(t\,|\,t-1) - \tilde{y}(t\,|\,t-1)\end{aligned}$$

On the other hand, $\widehat{\mathcal{M}}(\vartheta)$ and $\widetilde{\mathcal{M}}(\vartheta)$ are predictors and hence:

- $\hat{y}(t\,|\,t-1)$ depends on $y(t-1), y(t-2), \ldots$
- $\tilde{y}(t\,|\,t-1)$ depends on $y(t-1), y(t-2), \ldots$

Therefore $\hat{y}(t\,|\,t-1) - \tilde{y}(t\,|\,t-1)$ is uncorrelated with $\xi(t)$ and hence

$$\begin{aligned}\operatorname{var}[\tilde{\varepsilon}(t)] &= \operatorname{var}[\xi(t) + \hat{y}(t\,|\,t-1) - \tilde{y}(t\,|\,t-1)] \\ &= \operatorname{var}[\xi(t)] + \operatorname{var}[\hat{y}(t\,|\,t-1) - \tilde{y}(t\,|\,t-1)] \\ &\geq \operatorname{var}[\xi(t)] = \operatorname{var}[\hat{\varepsilon}(t)]\end{aligned}$$

which **contradicts** the assumption $\operatorname{var}[\tilde{\varepsilon}(t)] < \operatorname{var}[\hat{\varepsilon}(t)]$ hence proving that $\widehat{\mathcal{M}}(\vartheta)$ is optimal.

Models and Predictors (cont.)

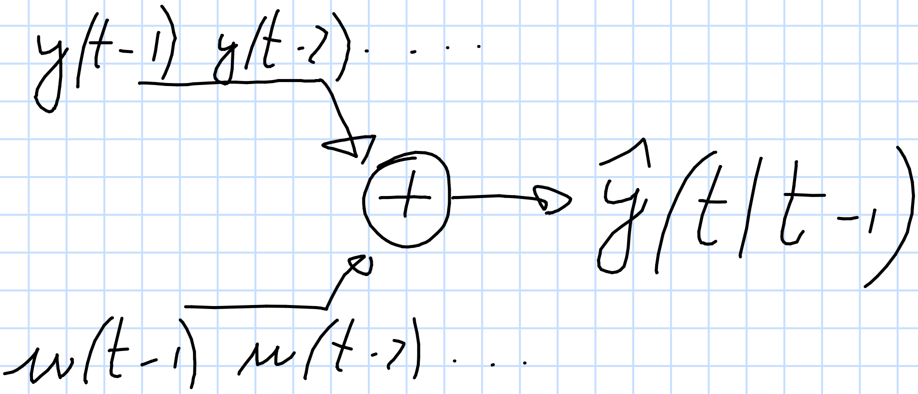
Summing up:

The model and its associated predictor

$$\mathcal{M}(\vartheta): \quad y(t) = G(z)\,u(t-1) + W(z)\,\xi(t)$$

$$\Longrightarrow \quad \widehat{\mathcal{M}}(\vartheta): \quad \hat{y}(t\,|\,t-1) = \left[1 - \frac{1}{W(z)}\right] y(t) + \frac{G(z)}{W(z)}\,u(t-1)$$

$\widehat{\mathcal{M}}(\vartheta)$ is called **model in prediction form**.


y(t-1) y(t-2) . . . .
+
ŷ(t|t-1)
u(t-1) u(t-2) . . .

Predictors for ARX Models

$$\mathcal{M}(\vartheta): \quad A(z)\,y(t) = B(z)\,u(t-1) + \xi(t)$$

$$1 - a_1 z^{-1} - a_2 z^{-2} \dots - a_n z^{-n}$$

$$\Longrightarrow \quad G(z) = \frac{B(z)}{A(z)} \qquad W(z) = \frac{1}{A(z)} \qquad \vartheta = \begin{array}{cl} \left[\begin{array}{c} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{array}\right] & \begin{array}{l} A(z) \\ \\ \\ B \\ \\ \end{array} \end{array}$$

$$\frac{B(z)}{z A(z)} \cdot u(t)$$

Then:

$$\begin{aligned} \hat{y}(t\,|\,t-1) &= \left[1 - \frac{1}{W(z)}\right] y(t) + \boxed{\frac{G(z)}{W(z)}\,u(t-1)} \\ &= [1 - A(z)]\; y(t) + B(z)\,u(t-1) \\ &= a_1\,y(t-1) + a_2\,y(t-2) + \cdots + a_n\,y(t-n) \\ &\quad + b_1\,u(t-1) + b_2\,u(t-2) + \cdots + b_n\,u(t-n) \end{aligned}$$

Observe that $\hat{y}(t\,|\,t-1)$ does not depend on its past values, that is, **the predictor is not dynamic and hence it is always stable**

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