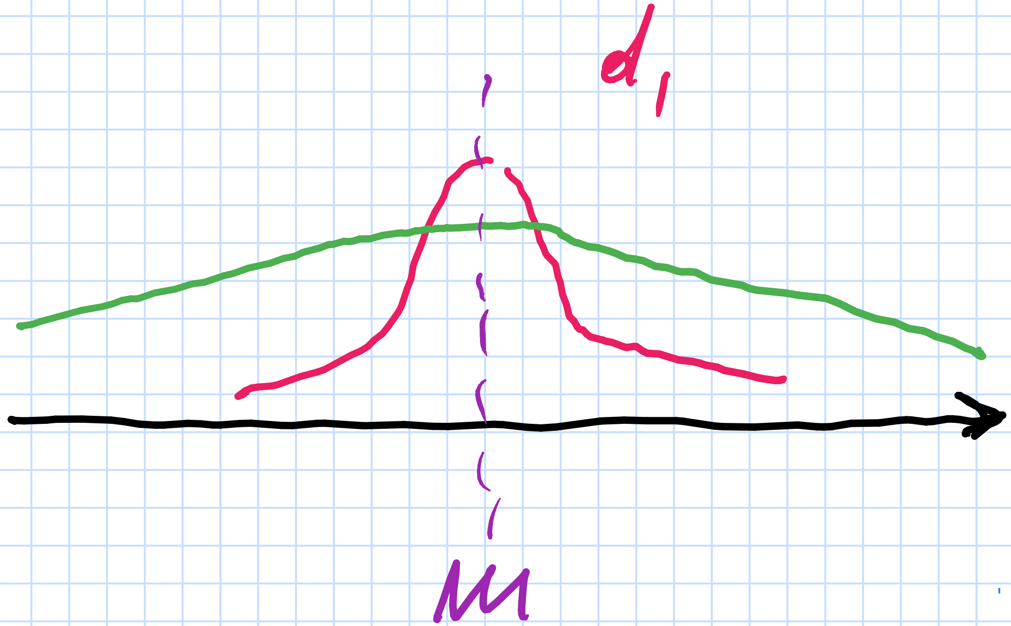
$\boxed{\text{Es}}$ 2 v.a. indipendenti, gaussiane d_1, d_2

$d_1 \sim \mathcal{G}(m, 2)$

$d_2 \sim \mathcal{G}(m, 4)$



Stimare m con una osservazione di d_1 ed d_2

$$\hat{m} = \alpha d_1 + \beta d_2$$

① NO BIAS $\quad E(\hat{\mu}) = \mu$

$$E[\hat{\mu}] = \alpha E[d_1] + \beta E[d_2]$$

$$= (\alpha + \beta)\mu$$

NO BIAS $\Longleftrightarrow$ $\alpha + \beta = 1$

② varianza minima

$$\hat{m} = \alpha d_1 + \beta d_2$$

$$\alpha + \beta = 1$$

$$\operatorname{var} \hat{m} = E\left[(\hat{m} - m)^2\right]$$

$$= E\left[(\alpha d_1 + \beta d_2 - \underbrace{1}_{\alpha+\beta} \cdot m)^2\right]$$

$$= E\left[\left(\alpha(d_1 - m) + \beta(d_2 - m)\right)^2\right] = \rightarrow$$

$\rightarrow \alpha^2 E\left[(d_1 - \mu)^2\right] + \beta^2 E\left[(d_2 - \mu)^2\right]$

$\hookrightarrow$ tutti gli altri termini valgono 0

$= \alpha^2 \cdot 2 + \beta^2 4 = 2\alpha^2 + 4\beta^2$

$\alpha + \beta = 1 \rightarrow \beta = 1 - \alpha$

$\mathrm{var}\,\hat{\mu} = 6\alpha^2 - 8\alpha + 4$

$$\operatorname{var} \hat{\mu} = 6\alpha^2 - 8\alpha + 4$$

$$\frac{d}{d\alpha}\left(\operatorname{var} \hat{\mu}\right) = 0 \longrightarrow 12\alpha - 8 = 0$$

$$\begin{cases} \alpha = 2/3 \\ \beta = 1/3 \end{cases}$$

$$\hat{\mu} = \frac{2}{3} d_1 + \frac{1}{3} d_2 \qquad \operatorname{var} \hat{\mu} = 4/3$$

Es ② consideriamo 2 v.a. x, w

$E(x) = 0$ $\quad E(w) = 0$ $\quad$ scorrelate

$\sigma_x^2 = 1$ $\quad \sigma_w^2 = 1$ $\quad E(xw) = 0$

Vorrei conoscere il valore attuale di x all'istante t (faccio un esperimento) $\to$ x non accessibile. Posso osservare un'altra v.a.

$$y = x + w$$

$\hat{x} = ?$ stimatore lineare ottimo

$$\hat{\gamma} = \frac{\phi_{\gamma d}}{\phi_{dd}} \, d$$

$\sigma_{\gamma d}$

σ_d^2

osservazione

$d \longleftrightarrow y$

$\gamma \longleftrightarrow x$

$\phi_{dd} \longleftrightarrow \sigma_y^2$!!!

$\phi_{\gamma d} \longleftrightarrow \sigma_{xy}$!!!

$$E(y) = ?$$

$$y = x + w$$

$$\longrightarrow E(y) = E(x) + E(w) = 0$$

$$\sigma_{xw} = 0$$

$$\sigma_y^2 = \sigma_x^2 + \sigma_w^2 = 2$$

$$\gamma_d = \sigma_{xy} = E(xy) \qquad y = x + w$$

$$E(xy) = E\left[x(x+w)\right] = \longrightarrow$$

$$E(xy) = E\left[x(x+w)\right] = \underbrace{E(x^2)}_{\sigma_x^2} + \underbrace{E(xw)}_{0}$$

$$= 1 = \sigma_{xy}$$

$$\hat{x} = \frac{\sigma_{xy}}{\sigma_{yy}} \cdot y = \frac{1}{2} y$$

$$\mathrm{var}(\theta - \hat{\theta}) = \sigma_{\theta\theta} - \frac{\sigma_{\theta d}^2}{\sigma_{dd}}$$

Supponiamo che le v.a. siano x, w, z IND.
e di poter fare 2 osservazioni differenti

$x \rightarrow E(x) = 0$ $\quad$ $\sigma_x^2 = 1$

$w \rightarrow E(w) = 0$ $\quad$ $\sigma_w^2 = 1$

$z \rightarrow E(z) = 0$ $\quad$ $\sigma_z^2 = d^2$

$$\begin{cases} y_1 = x + w \\ y_2 = x + z \end{cases}$$

$$\hat{y} = \Lambda_{yd} \cdot \Lambda_{dd}^{-1} \cdot d$$

$$\hat{x} = \Lambda_{xd} \cdot \Lambda_{dd}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Lambda_{dd} = \begin{bmatrix} \sigma_{y_1}^2 & \sigma_{y_1 y_2} \\ \sigma_{y_2 y_1} & \sigma_{y_2}^2 \end{bmatrix}$$

$$\sigma_{y}^2 = 2$$

$$\sigma_{y_2}^2 = 1 + d^2$$

$$\sigma_{y_1 y_2} = E[y_1 y_2] = E[(x+w)(x+z)]$$

$$= E(x^2) + \cancel{E(xw)} + \cancel{E(xz)} + \cancel{E(wz)}$$

$$\phantom{=E(x^2)+{}} 0 \qquad\quad 0 \qquad\quad 0$$

$$= 1$$

$$\Lambda_{old} = \begin{bmatrix} 2 & 1 \\ 1 & 1+d^2 \end{bmatrix}$$

$$\Lambda_{dd}^{-1} = \begin{bmatrix} \dfrac{d^2+1}{2d^2+1} & -\dfrac{1}{2d^2+1} \\[2ex] -\dfrac{1}{2d^2+1} & \dfrac{2}{2d^2+1} \end{bmatrix}$$

$$\Lambda_{d\theta} = \begin{bmatrix} E(y_1 x) \\ E(y_2 x) \end{bmatrix} \qquad \Lambda_{\theta d} = \Lambda_{d\theta}^{T}$$

$$E(y_1 x) = E[(x+w)x] = 1$$

$$E(y_2 x) = E[(x+z)x] = 1$$

$$\Lambda_{d\sigma} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad \Lambda_{\sigma d} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\hat{x} = \Lambda_{\sigma d} \cdot \Lambda_{dd}^{-1} \cdot d = \begin{bmatrix} 1 & 1 \end{bmatrix} \Lambda_{dd}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2\alpha^2+1} \left(y_2 + \alpha^2 y_1 \right)$$

se $\alpha^2 \gg 1 \implies \hat{x} \to \frac{1}{2} y_1$

se $\alpha^2 \ll 1 \implies \hat{x} \to y_2$

E se $y_2 = z$?

$$\begin{cases} y_1 = x + w \\ y_2 = z \end{cases}$$

$$E(z) = 0$$

$$\sigma_z^2 = 1$$

$$E(xw) = 0$$

$$E(xz) = 0$$

$$E(wz) = 0$$

$$\Lambda_{dd} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Lambda_{d\sigma} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{x} = \Lambda_{dd}^{\,} \cdot \Lambda_{dd}^{-1} \cdot d$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2} y_1$$

$$\mathrm{var}(x-\hat{x}) = \sigma_x^2 - \Lambda_{\sigma d}\Lambda_{dd}^{-1}\Lambda_{d\sigma}$$

$$= 1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 1/2$$

Es Processo stocastico descritto da

$$\begin{cases} y(t) = a^{\circ} u(t) + v(t) \\ v(t) = e(t) + \frac{1}{2} e(t-1) \end{cases}$$

$\mathcal{S}$

$e \sim WN(0,1)$

$u(t) \equiv 1 \quad \forall t$ (deterministico)

$\mathcal{M}$ $y(t) = a\,u(t) + \eta(t) \quad \eta \sim WN(0, \sigma^2)$

Si vuole stimare a usando MEP LS.

$$\hat{a}(N) \xrightarrow[N \to +\infty]{} ?$$

MEP $\longrightarrow$ $\hat{y}(t|t-1) = ?$

predittore ad 1 passo

$$y(t) = a\,u(t) + \cancel{\eta(t)}$$

$$\hat{y}(t|t-1) = a\,u(t)$$

errore di predizione

$$\varepsilon(t) = y(t) - \hat{y}(t|t-1)$$

funzionale di costo

$$J_N(\alpha) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t)$$

per $N \to \infty$

$$J_N(\alpha) \xrightarrow[N \to +\infty]{} E\left[\varepsilon^2(t)\right] = \bar{J}(\alpha)$$

$$J(a) = E\left[\varepsilon^2(t)\right] = ?$$

$$\varepsilon(t) = y(t) - \hat{y}(t|t-1)$$

$$= \underbrace{a^{\circ} u(t) + v(t)}_{y(t)} - \overbrace{a\, u(t)}^{\hat{y}(t|t-1)}$$

$$= (a^{\circ} - a)\, \overbrace{u(t)}^{\equiv 1} + v(t)$$

$$\varepsilon(t) = (a^\circ - a) + v(t)$$

$$J(a) = E\left[\varepsilon^2(t)\right] =$$

$$= E\left[\left((a^\circ - a) + v(t)\right)^2\right] =$$

$$= (a^\circ - a)^2 + E\left[v^2(t)\right] +$$

$$+ 2(a^\circ - a)\, E\left[v(t)\right]$$

$$E\left[v^2(t)\right] =$$

$$v(t) = e(t) + \frac{1}{2} e(t-1)$$

$$e \sim WN(0, 1)$$

$MA(1)$

$$E(v) = 0$$

$$\operatorname{var} v = \left[1^2 + \left(\frac{1}{2}\right)^2\right] 1$$

$$= 5/4$$

$$\bar{J}(a) = (a^0 - a)^2 + \overbrace{E[v^2]}^{5/4} + 2(a^0 - a)\cancel{E(v)}_{0}$$

$$= (a^0 - a)^2 + 5/4$$

$$\min \bar{J}(a) \longrightarrow \hat{a} = a^0$$

NB l'errore di predizione è MA(1)