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Robust Minimum-time Constrained Control of Nonlinear Discrete-Time Systems: New Results

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Abstract

This work is concerned with the analysis and the design of robust minimum-time control laws for nonlinear discrete-time systems with possibly non robustly controllable target sets. We will show that, given a Lipschitz nonlinear transition map with bounded control inputs, the reachability properties of the target set can be used to assess the existence of a robust positively controllable set which includes the target in its interior. This result will be exploited to formulate a robustified minimum-time control scheme capable to ensure the ultimate boundedness of the trajectories in presence of bounded uncertainties even if the target set is not robust positively controllable.

I. Introduction

The Minimum-time control problem consists in steering the state of a controlled system from an initial point $x_0 \in \mathbb{R}^n$ to a given closed set $\Xi \subset \mathbb{R}^n$ (the so-called "target" set) in minimum time.

The solution of the minimum-time problem is well-known in the case of linear systems with compact target sets (see [1], [2], [3], [4], [5], the survey paper [6] and the references therein),

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while further investigations are needed both to characterize the stability properties of nominal minimum-time control laws in a nonlinear setting, and to design minimum-time controllers robust with respect to unmodelled nonlinearities and unknown external disurbances (see [7] and [8], [9] for two robust formulations based on dynamic programming and invariant-set theory for linear systems). Indeed, since the mathematical models available for the control design are often subjected to uncertainty and the system may be affected by exogenous not measurable inputs (disturbances) which are not a-priori known, in practice the synthesis of the control scheme is performed with incomplete informations.

As for the linear case, it can be proven that if the target set is robust positively controllable (i.e., the target set can be rendered robust positively invariant by some control law veryfing the input constraints, [10]), then the nonlinear minimum-time control ensures the uniform boundedness of the closed-loop trajectories for a suitable set of initial conditions, [11], [12], [6]. In addition, we will show that the ultimate boundedness property can be preserved even if the target set is not one-step robust positively controllable, by suitably modifying the nominal minimum-time control law. As far as it is known to the authors, the problem of guaranteeing the boundedness of the trajectories by minimum-time cotrol with non controllable terminal sets has never been addressed in the current literature.

On the other side, the Minimum-Time control problem, in the discrete-time nonlinear context, is stricly connected to the Finite-Horizon Optimal Control Problem (FHOCP) with terminal set constraints, that is the optimization scheme which conventional Nonlinear Model Predictive Control (NMPC) relies on (see e.g. [13], [14], [15], [16] and [17] among the vast literature on the subject). The NMPC technique consists in solving the FHOCP repeatedly along system's trajectories, with respect to a sequence of control actions, and in applying to the controlled system only the first control move computed by each optimization. In the NMPC framework the terminal constraint is introduced with the only aim of providing robust stability guarantees, therefore it is usually chosen as an arbitrary robust positively controllable set, [18], [19], [20]. The inclusion of such a supplementary terminal condition introduces some conservatism and raises the additional issue of the recursive feasibility with respect to the new constraint (see [21], [22]). Conversely, in the minimum-time control setting the terminal set represents it by itself the objective of the control design. If the specified target set is not control invariant, then, to achieve closed-loop robustness, the minimum-time control law is usually computed by imposing a different terminal constraint, chosen as an invariant subset of the nominal target set. In this case, the finite-time reachability of the target set , as well as the ultimate boundedness of the trajectories, can be guaranteed by set-theoretic arguments (see [5]). Nonetheless, the contraction of the target set represents a conservative provision for achieving the robust trajectory boundedness and the finite-time reach of the target in absence of uncertainties. In particular, the robust stability properties of the modified minimum-time problem with a restricted terminal region are achieved at the cost of a smaller feasible region, that is, a smaller capture basin.

Exploiting some ideas originally conceived by the author in field of NMPC (see [22]), an alternative design procedure is therefore proposed in this work to reduce the conservatism of the conventionl minimum-time formulation, that allows to retain the original target set without restrictions. Moreover, when the target is robustly controllable, the devised methods guarantees the same robust performance (minimum reach-time and maximal admissible uncertainty for trajectory boundedness) of standard approaches.

The paper is organized as follows. In Section II we will introduce the notation and some preliminary technical result. In Section III the minimum-time control problem for nonlinear systems will be formalized and discussed. In Section IV the properties of the nominal minimum-time control, in dependence of its design parameters (with particular focus on the target set), will be analyzed. Finally, in Section V, a control scheme will be proposed to guarantee the boundedness of the trajectories despite the presence of bounded uncertainties and possibly non robust positively controllable target sets.

II. PRELIMINARIES

In the following, the notation that will be used throughout the paper will be introduced, together with the basic assumptions and the technical results that will be needed to derive the main theorems.

A. Notation

Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$. Given a signal s, let $\mathrm{s}_{[t_1,t_2]}$ be a sequence defined from time t_1 to time t_2. In order to simplify the notation, when it is inferrable from the context, the subscript of the sequence is omitted. The set of discrete-time sequences of s taking values in some subset $\Upsilon \subset \mathbb{R}^n$ is denoted by $\mathcal{M}_\Upsilon$. Moreover let us define $||\mathrm{s}|| \triangleq \sup_{k\geq 0}\{|s_k|\}$ and $||\mathrm{s}_{[t_1,t_2]}|| \triangleq \sup_{t_1\leq k\leq t_2}\{|s_k|\}$, where s_k denotes the value that the sequence s takes on in correspondence with the index k. Given a compact set $A \subseteq \mathbb{R}^n$, let ∂A denote the boundary of A. Given a vector $x \in \mathbb{R}^n$, $\mathrm{d}(x, A) \triangleq \inf\{|\xi - x|, \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to A, while $\Phi(x, A) \triangleq \{\, \mathrm{d}(x, \partial A)$ if $x \in A$, $-\mathrm{d}(x, \partial A)$ if $x \notin A \,\}$ denotes the signed distance function. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, $\mathrm{dist}(A, B) \triangleq \inf\{\mathrm{d}(\zeta, A), \zeta \in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, then the Pontryagin difference set C is defined as $C = A \backsim B \triangleq \{x \in \mathbb{R}^n : x + \xi \in A, \forall \xi \in B\}$, while the Minkowski sum set is defined as $S = A \oplus B \triangleq \{x \in \mathbb{R}^n : x = \xi + \eta,\ \xi \in A,\ \eta \in B\}$. Given a vector $\eta \in \mathbb{R}^n$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball centered in η and of radius ρ is denoted as $\mathcal{B}^n(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^n : |\xi - \eta| \leq \rho\}$. The shorthand $\mathcal{B}^n(\rho)$ is used when the ball is centered in the origin. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\gamma(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ ($\mathcal{K}$-function) if it is continuous, zero at zero, and strictly increasing.

B. Basic Assumptions and Definitions

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t\,,\ t \in \mathbb{Z}_{>0},\ x_0 = \bar{x} \tag{1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, u_t the control vector, subject to the constraint

$$u_t \in U \subset \mathbb{R}^m, \tag{2}$$

with U compact, and $d_t \in D \subset \mathbb{R}^n$, with D compact, a bounded additive transition uncertainty vector.

In stating and proving the preliminary technical lemmas, and with the aim of simplifying the derivation of the main results, let the function $\hat{f}(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$ verify the following assumption.

Assumption 1 (Lipschitz): The function $\hat{f}(x,u)$ is Lipschitz (L.) continuous w.r.t. $x \in \mathbb{R}^n$, uniformly in $u \in U$, with L. constant $L_{\hat{f}_x} \in \mathbb{R}_{\geq 0}$, that is, for all $x \in \mathbb{R}^n$ and $x^{'} \in \mathbb{R}^n$

$$|\hat{f}(x,u) - \hat{f}(x^{'},u)| \leq L_{\hat{f}_x}|x - x^{'}|, \forall u \in U.$$

□

Moreover, to prove some results we will also pose the following assumption.

Assumption 2 (Local Uniform Continuity w.r.t. u): For any $x \in \mathbb{R}^n$ the function $\hat{f}(x,u)$ is uniformly continuous w.r.t. $u \in U$. That is, for any $u \in U$ and any $u^{'} \in U$

$$|\hat{f}(x,u) - \hat{f}(x,u^{'})| \leq \eta_u(|u - u^{'}|), \forall x \in \mathbb{R}^n.$$

where $\eta_u(\cdot)$ is a $\mathcal{K}$-function. □

Definition 2.1 (Controllability set to Ξ): Given a map $\hat{f}(x,u) : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ compact, and a set $\Xi \subset \mathbb{R}^n$, the (one-step) controllability set to Ξ, ($\mathcal{C}_1(\Xi)$) is given by

$$\mathcal{C}_1(\Xi) \triangleq \left\{ x_0 \in \mathbb{R}^n | \exists\, u_{x_0} \in U : \hat{f}(x_0, u_{x_0}) \in \Xi \right\}. \tag{3}$$

□

Definition 2.2 (Predecessor set of Ξ): Given the map $\hat{g}(x):\mathbb{R}^n \to \mathbb{R}^n$ and a set $\Xi \subset \mathbb{R}^n$, the (one-step) predecessor of Ξ, ($\mathcal{P}_1(\Xi)$) is given by

$$\mathcal{P}_1(\Xi) \triangleq \{x_0 \in \mathbb{R}^n | \hat{g}(x_0) \in \Xi\}. \tag{4}$$

□

Definition 2.3 (i-steps Controllability Set to Ξ): Given a map $\hat{f}(x,u):\mathbb{R}^n \times U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ compact, and a set $\Xi \subset \mathbb{R}^n$, the i-steps controllability set to Ξ, ($\mathcal{C}_i(\Xi)$) is given by

$$\mathcal{C}_i(\Xi) \triangleq \left\{x_0 \in \mathbb{R}^n | \exists\, \mathbf{u}_{x_0} \in U^i : \hat{x}(i,x,\mathbf{u}_{x_0}) \in \Xi\right\}. \tag{5}$$

that is, $\mathcal{C}_i(\Xi)$ is the set o initial states $x_0 \in \mathbb{R}^n$ from which Ξ can be reached in exact i steps. □

Definition 2.4 (i-steps Predecessor of Ξ): Given a map $\hat{g}(x):\mathbb{R}^n \to \mathbb{R}^n$ and a set $\Xi \subset \mathbb{R}^n$, the i-steps predecessor of Ξ, ($\mathcal{P}_i(\Xi)$) is given by

$$\mathcal{P}_i(\Xi) \triangleq \left\{x_0 \in \mathbb{R}^n | \hat{g}^i(x_0) \in \Xi\right\}. \tag{6}$$

□

Definition 2.5 (i-steps Capture Basin to Ξ): Given a map $\hat{f}(x,u):\mathbb{R}^n \times U \to \mathbb{R}^n$, with $U \subset \mathbb{R}^m$ compact, and a set $\Xi \subset \mathbb{R}^n$, the i-steps capture basin to Ξ, ($\mathrm{Capt}_i(\Xi)$) is given by

$$\mathrm{Capt}_i(\Xi) \triangleq \bigcup_{j=1}^{i} \mathcal{C}_i(\Xi). \tag{7}$$

that is, $\mathrm{Capt}_i(\Xi)$ is the set o initial states $x_0 \in \mathbb{R}^n$ such that Ξ is reached in at most i steps (i.e., $\exists\, \mathbf{u}_{x_0} \in U^j : \hat{x}(j,x_0,\mathbf{u}_{x_0}) \in \Xi$ for at least one $j \in [1,\ldots,i-1]$, before possibly leaving Ξ. □

Moreover, the following property holds for controllability sets.

Proposition 2.1: Given two sets $\Xi_1 \subset \mathbb{R}^n$ and $\Xi_2 \subset \mathbb{R}^n$, then $\mathcal{C}_1(\Xi_1 \cup \Xi_2) = \mathcal{C}_1(\Xi_1) \cup \mathcal{C}_1(\Xi_2)$.

□

Proof: From Definition 2.1 we have that

$$\begin{aligned}\mathcal{C}_1(\Xi_1 \cup \Xi_2) &\triangleq \{x \in \mathbb{R}^n | \, \exists u_x : \hat{f}(x, u_x) \in (\Xi_1 \cup \Xi_2)\} \\ &= \{\xi \in \mathbb{R}^n | \, \exists u_\xi : \hat{f}(\xi, u_\xi) \in (\Xi_1)\} \\ &\quad \cup \{x \in \mathbb{R}^n | \, \exists u_x : \hat{f}(x, u_x) \in (\Xi_2)\} \\ &= \mathcal{C}_1(\Xi_1) \cup \mathcal{C}_1(\Xi_2),\end{aligned}$$

thus proving the statement. ■

Definition 2.6 (RC, d_1-RC): A compact set $\Xi \subset \mathbb{R}^n$ is Robustly Controllable (RC) under the map $\hat{f}(x, u)$, with $u \in U$, if $\Xi \subseteq \mathcal{C}_1 (\text{int}(\Xi))$.

A compact set $\Xi \subset \mathbb{R}^n$ is Robustly Controllable in one-step w.r.t. additive perturbations $d \in \mathcal{B}^n(d_1)$ (d_1-RC) if $(\Xi \backsim \mathcal{B}^n (d_1))$ is not empty and $\Xi \subseteq \mathcal{C}_1(\Xi \backsim \mathcal{B}^n (d_1))$. □

Definition 2.7 (RPI): A compact set $\Xi \subset \mathbb{R}^n$ is Robust Positively Invariant (RPI) under the map $\hat{g}(x)$, if $\Xi \subseteq \mathcal{P}_1 (\text{int}(\Xi))$. □

Definition 2.8 (quasi-RPI): A compact set $\Xi \subset \mathbb{R}^n$ is quasi-Robust Positively Invariant (RPI) under the map $\hat{g}(x)$ with maximum return-time N, if $\Xi \subseteq \bigcup_{i=1}^{N} \mathcal{P}_i (\text{int}(\Xi))$ for some finite $N \in \mathbb{Z}_{>0}$; that is, for any $x_0 \in \bigcup_{i=1}^{N} \mathcal{P}_i (\text{int}(\Xi))$, $\exists i \in \{1, \dots, N\} : \hat{g}^i(x_0) \in \text{int}(\Xi)$. □

Next, some results concerning the properties of robustly controllable sets under Lipschitz maps are given. These intermediate results will be used in Section IV to prove the main contribution of the present work.

C. Preliminary Technical Results

In stating and deriving the following technical lemmas, let the nominal system's transition map $\hat{f}$ verify Assumption 1.

Lemma 2.1 (Technical): Given two compact sets $\Xi_1 \subset \mathbb{R}^n$, $\Xi_2 \subset \mathbb{R}^n$ and a positive scalar $d \in \mathbb{R}_{>0}$, if the following three conditions hold together: $i)$ $\Xi_1 \subseteq \mathcal{C}_1(\Xi_2)$, $ii)$ $\Xi_2 \subset \Xi_1$ and $iii)$ $\text{dist}(\mathbb{R}^n \backslash \Xi_1, \Xi_2) \geq d$, then Ξ_1 is d-RC. □

Proof: The two conditions $ii)$ and $iii)$ together imply that

$$\Xi_2 \subseteq \Xi_1 \backsim \mathcal{B}^n(d). \tag{8}$$

Finally, from the inclusion $\Xi_1 \subseteq \mathcal{C}_1(\Xi_2)$ and (8) it follows that

$$\Xi_1 \subseteq \mathcal{C}_1\Big(\Xi_1 \backsim \mathcal{B}^n(d)\Big),$$

which proves the statement of the Lemma. ■

Lemma 2.2 (Technical): Given a compact set $\Xi \subset \mathbb{R}^n$, assume that $\mathcal{C}_1(\Xi)$ is non-empty. Then, for any arbitrary $\eta \in \mathbb{R}_{\geq 0}$ it holds that:

$$\forall x \in \Xi \oplus \mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta\right),\ \exists\, u_x \in U\ :\ \hat{f}(x, u_x) \in \Xi \oplus \mathcal{B}^n(\eta). \tag{9}$$

□

Proof: Consider the set $\mathcal{C}_1(\Xi) \oplus \mathcal{B}(L_{\hat{f}_x}^{-1}\eta)$. It holds that $\forall x \in \Big(\mathcal{C}_1(\Xi) \oplus \mathcal{B}(L_{\hat{f}_x^{-1}}\eta)\Big)$, $\exists\, \xi_x \in \mathcal{C}_1(\Xi)$ such that

$$|x - \xi_x| \leq L_{\hat{f}_x}^{-1}\eta. \tag{10}$$

The inclusion $\xi_x \in \mathcal{C}_1(\Xi)$ implies that $\exists\, u_{\xi_x} \in U\ :\ \hat{f}(\xi_x, u_{\xi_x}) \in \Xi$. Since $\hat{f}$ is Lipschitz, then $|\hat{f}(x, u_{\xi_x}) - \hat{f}(\xi_x, u_{\xi_x})| \leq L_{\hat{f}_x}|x - \xi_x|$. From (10), it follows that

$$|\hat{f}(x, u_{\xi_x}) - \hat{f}(\xi_x, u_{\xi_x})| \leq \eta. \tag{11}$$

which finally yields to the statement of the Lemma. ■

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Lemma 2.3 (Technical): Given a compact set $\Xi \subset \mathbb{R}^n$ and positive scalar $\rho \in \mathbb{R}_{>0}$, if Ξ is ρ-RC, then $\mathcal{C}_1(\Xi)$ is $(L_{\hat{f}_x}^{-1}\rho)$-RC. □

Proof: Consider the set $\Xi \oplus \mathcal{B}(L_{\hat{f}_x}^{-1}\rho)$. It holds that $\forall \xi \in (\Xi \oplus \mathcal{B}(L_{\hat{f}_x}^{-1}\rho))$, $\exists x_\xi \in \Xi$ and $\exists u_{x_\xi} \in U$ such that $|\xi - x_\xi| \leq L_{\hat{f}_x}^{-1}\rho$ and $\hat{f}(x_\xi, u_{x_\xi}) \in (\Xi \backsim \mathcal{B}^n(\rho))$. Let us pose $u_\xi = u_{x_\xi}$. Then, in view of Assumption 1, it holds that $\hat{f}(\xi, u_\xi) \in \Xi, \forall \xi \in (\Xi \oplus \mathcal{B}(L_{\hat{f}_x}^{-1}\rho))$, which implies that

$\mathcal{C}_1(\Xi)\supseteq(\Xi\oplus\mathcal{B}(L_{\hat{f}_x}^{-1}\rho))$ and that $\text{dist}(\mathbb{R}^n\backslash\mathcal{C}_1(\Xi),\Xi)\geq L_{\hat{f}_x}^{-1}\rho$. Finally, thanks to Lemma 2.1, the statement follows. ■

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The following technical result will establish the invariant properties of the N-steps controllability set $\mathcal{C}_N(\Xi)$ of a given ρ-RC set Ξ. Moreover, an inner (conservative) approximation of $\mathcal{C}_N(\Xi)$, containing Ξ in its interior, will be provided.

Lemma 2.4 (Technical): Given a compact set $\Xi\subset\mathbb{R}^n$, a finite integer $N\in\mathbb{Z}_{>0}$ and a positive scalar $\rho\in\mathbb{R}_{>0}$, if Ξ is ρ-RC, then

i) $\mathcal{C}_N(\Xi)$ is $(L_{\hat{f}_x}^{-N}\rho)$-RC.

ii) $\mathcal{C}_N(\Xi)\supseteq\Xi\oplus\mathcal{B}^n\left(\dfrac{1-L_{\hat{f}_x}^{-N}}{L_{\hat{f}_x}-1}\rho\right)$

□

Proof: Applying recursively Lemma 2.3, it holds that

$$\text{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(\Xi),\mathcal{C}_{i-1}(\Xi))\geq L_{\hat{f}_x}^{-i}\rho,\forall i\in\mathbb{Z}_{>0},\tag{12}$$

with $\mathcal{C}_0(\Xi)=\Xi$. Therefore we obtain

$$\text{dist}(\mathbb{R}^n\backslash\mathcal{C}_N(\Xi),\Xi)\geq\rho\sum_{i=1}^{N}L_{\hat{f}_x}^{-i}=\frac{1-L_{\hat{f}_x}^{-N}}{L_{\hat{f}_x}-1}\rho.\tag{13}$$

Finally, the statement follows from inequality (13) and Lemma 2.1. ■

The following important result, that will play a key role in characterizing the robust stability properties of nonlinear minimum-time control laws, can now be proven. The reader can refer to Figure 1 for a schematization of the sets involved in the statement and in the proof of the forthcoming theorem.

Theorem 2.1 (N-steps Reachability Implication): Given a compact set $\Xi\subset\mathbb{R}^n$ and map $\hat{f}(x,u)$ verifying Assumption 1 and subject to (2), if the following inclusion holds for a finite integer $N\in\mathbb{Z}_{>0}$ and for a positive scalar $\rho_N\in\mathbb{R}_{>0}$:

$$\Xi\subseteq\text{Capt}_N(\Xi)\backsim\mathcal{B}^n(\rho_N),\tag{14}$$

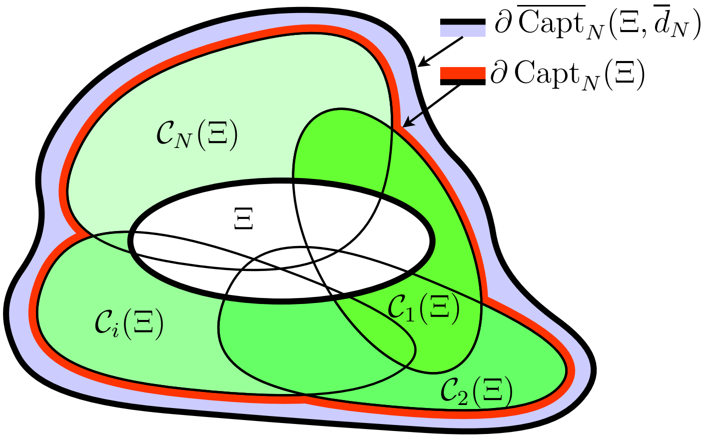

Fig. 1. Scheme of the sets involved in the statement and proof of Theorem 2.1. $\mathrm{Capt}_N(\Xi)$ denotes the N-steps capture basin of Ξ, while $\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N)$ is an extension of the nominal capture basin (see (15)) that can be proven to be robust positively controllable. That is, there exists a control law, compliant with the input constraints, that renders $\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N)$ RPI.

(i.e., Ξ is reachable in at most N steps from a set containing Ξ in its interior under the nominal map $\hat{f}(x,u)$), then the set

$$\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N) \triangleq \mathcal{C}_N(\Xi) \cup \left(\bigcup_{i=1}^{N-2} \left[\mathcal{C}_{i+1}(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right)\right]\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right) \tag{15}$$

is $\overline{d}_N$-RC, with η_i positive scalars depending on $\overline{d}_N$ according to the recursion

$$\begin{aligned} \overline{d}_N &\triangleq \frac{L_{\hat{f}_x}-1}{L_{\hat{f}_x}^N-1}\rho_N, \\ \eta_0 &= \rho_N - \overline{d}_N, \end{aligned} \tag{16}$$

and

$$\eta_i \;= L_{\hat{f}_x}^{-1}\eta_{i-1} - \overline{d}_N,\ \forall i \in \{1,\dots N-2\}.$$

$\square$

Proof: Assume, at this stage, that the condition (14) holds for $N=1$. In this case the set $\overline{\mathrm{Capt}}_1(\Xi,d_1) = \mathcal{C}_1(\Xi)$ is $\overline{d}_1$-RC by definition, with $\overline{d}_1 = \rho_1$.

Now, let us consider $N=2$ and $\rho_2 = \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_2(\Xi), \Xi)$. Being $\Xi \subseteq (\mathrm{Capt}_2(\Xi) \backsim \mathcal{B}^n(\rho_2))$, it follows that,

$$\forall x\in\mathcal{C}_1(\Xi),\ \exists\, u_x\in U\colon \hat{f}(x,u_x)\in(\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2)),\tag{17}$$

which, in view of Lemma 2.2, implies that

$$\forall x\in\mathcal{C}_1(\Xi)\oplus\mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right),\ \exists\, u_x\in U\colon \hat{f}(x,u_x)\in\left(\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2-\eta_0)\right),\tag{18}$$

for all $\eta_0\in[0,\rho_2]$. Moreover, from the definition of $\mathcal{C}_2(\Xi)$, we recall that

$$\forall x\in\mathcal{C}_2(\Xi),\ \exists\, u_x\in U\colon \hat{f}(x,u_x)\in\mathcal{C}_1(\Xi),\tag{19}$$

Now, from (18),(19) and Property 2.1 it follows that

$$\forall x\in\mathcal{C}_2(\Xi)\cup\left[\mathcal{C}_1(\Xi)\oplus\mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right],\ \exists\, u_x\in U\colon \hat{f}(x,u_x)\in\mathcal{C}_1(\Xi)\cup\left[\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2-\eta_0)\right],\tag{20}$$

that is

$$\mathcal{C}_2(\Xi)\cup\left[\mathcal{C}_1(\Xi)\oplus\mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right]\subseteq\mathcal{C}_1\Big(\mathcal{C}_1(\Xi)\cup\left[\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2-\eta_0)\right]\Big),\tag{21}$$

Moreover, if $\eta_0\in(0,\rho_2)$, then it holds that

$$\mathcal{C}_1(\Xi)\cup\left[\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2-\eta_0)\right]\subset\mathcal{C}_2(\Xi)\cup\left[\mathcal{C}_1(\Xi)\oplus\mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right]\tag{22}$$

Finally, from the topology of the sets involved in the underlying analysis it follows that

$$\mathrm{dist}\Big(\mathbb{R}^n\backslash\left(\mathcal{C}_2(\Xi)\cup\left[\mathcal{C}_1\oplus\mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right]\right),\ \mathcal{C}_1(\Xi)\cup\left[\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2-\eta_0)\right]\Big)\geq\min\left\{L_{\hat{f}_x}^{-1}\eta_0,\rho_2-\eta_0\right\}.\tag{23}$$

Now, consider the case $\rho_2-\eta_0=L_{\hat{f}_x}^{-1}\eta_0$. Posing

$$\hat{\eta_0}\triangleq\left(1+L_{\hat{f}_x}^{-1}\right)^{-1}\rho_2 \quad=\quad \frac{L_{\hat{f}_x}}{L_{\hat{f}_x}+1}\rho_2$$

then inequality (23) holds with positive right-hand side:

$$\mathrm{dist}\Big(\mathbb{R}^n\backslash\left(\mathcal{C}_2(\Xi)\cup\left[\mathcal{C}_1\oplus\mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right]\right),\ \mathcal{C}_1(\Xi)\cup\left[\mathrm{Capt}_2(\Xi)\backsim\mathcal{B}^n(\rho_2-\eta_0)\right]\Big)\geq\rho_2-\hat{\eta_0}=\frac{1}{1+L_{\hat{f}_x}}\rho_2.\tag{24}$$

In view of Lemma 2.1, (21), (22) and (24) together imply that the set $\overline{\mathrm{Capt}}_2(\Xi, d_2) = \mathcal{C}_2(\Xi) \cup \left[\mathcal{C}_1 \oplus \mathcal{B}^n\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right]$ is $\overline{d}_2$-RC, with

$$\overline{d}_2 = \frac{1}{1 + L_{\hat{f}_x}}\rho_2.$$

Now let us consider the case $N = 3$ and $\rho_3 = \mathrm{dist}\left(\mathbb{R}^n \backslash \mathrm{Capt}_3(\Xi), Xi\right)$. Being $\Xi \subseteq (\mathrm{Capt}_3(\Xi) \backsim \mathcal{B}^n(\delta_3))$, it follows that

$$\forall x \in \mathcal{C}_1(\Xi),\ \exists\, u_x \in U\colon \hat{f}(x, u_x) \in (\mathrm{Capt}_3(\Xi) \backsim \mathcal{B}^n(\rho_3)), \tag{25}$$

which, in view of Lemma 2.2, implies that

$$\forall x \in \mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right),\ \exists\, u_x \in U\colon \hat{f}(x, u_x) \in \left(\mathrm{Capt}_3(\Xi) \backsim \mathcal{B}^n(\rho_3 - \eta_0)\right), \tag{26}$$

for all $\eta_0 \in [0, \rho_3]$. Moreover, from the definition of $\mathcal{C}_2(\Xi)$ (see (19)) and by exploiting Lemma 2.2, we have that

$$\forall x \in \mathcal{C}_2(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta\right),\ \exists\, u_x \in U\colon \hat{f}(x, u_x) \in \mathcal{C}_1(\Xi) \oplus \mathcal{B}(\eta), \tag{27}$$

for all $\eta \in \mathbb{R}_{\geq 0}$. Finally, from the definition of $\mathcal{C}_3(\Xi)$, we have that

$$\forall x \in \mathcal{C}_3(\Xi),\ \exists\, u_x \in U\colon \hat{f}(x, u_x) \in \mathcal{C}_2(\Xi), \tag{28}$$

From (26), (27), (28) and thanks to Property 2.1, then the following inclusion holds

$$\begin{aligned}
&\mathcal{C}_3(\Xi) \cup \left(\mathcal{C}_2(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta\right)\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right) \\
&\quad \subseteq \mathcal{C}_1\left(\mathcal{C}_2(\Xi) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}(\eta)\right) \cup \left(\mathrm{Capt}_3(\Xi) \backsim \mathcal{B}^n(\rho_3 - \eta_0)\right)\right)
\end{aligned} \tag{29}$$

Moreover, if $\eta > 0$, $\eta < {L_{\hat{f}_x}}^{-1}\eta_0$ and $\eta_0 \in (0, \rho_3)$, then the following inclusion holds

$$\begin{aligned}
&\mathcal{C}_2(\Xi) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}(\eta)\right) \cup \left(\mathrm{Capt}_3(\Xi) \backsim \mathcal{B}^n(\rho_3 - \eta_0)\right) \\
&\quad \subset \mathcal{C}_3(\Xi) \cup \left(\mathcal{C}_2(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta\right)\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right)
\end{aligned} \tag{30}$$

Finally, from the topology of the sets involved it turns out that

$$\begin{aligned}
&\mathrm{dist}\Bigg(\mathbb{R}^n \backslash \left[\mathcal{C}_3(\Xi) \cup \left(\mathcal{C}_2(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta\right)\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right)\right], \\
&\qquad \left[\mathcal{C}_2(\Xi) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}(\eta)\right) \cup \left(\mathrm{Capt}_3(\Xi) \backsim \mathcal{B}^n(\rho_3 - \eta_0)\right)\right]\Bigg) \\
&\qquad \geq \min\left\{\rho_3 - \eta_0, L_{\hat{f}_x}^{-1}\eta_0 - \eta, L_{\hat{f}_x}^{-1}\eta\right\}
\end{aligned} \tag{31}$$

Choosing η_0 and η such that all the three terms in the right-hand side of (31) are equal, then inequality (31) holds with positive right-hand side.

Thanks to Lemma 2.1, then (29), (30) and (31) together imply that the set $\overline{\mathrm{Capt}}_3(\Xi, d_3) = \mathcal{C}_3(\Xi) \cup \left(\mathcal{C}_2(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta\right)\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right)$ is $\overline{d}_3$-RC,with $\overline{d}_3 = (L_{\hat{f}_x}^2 + L_{\hat{f}_x} + 1)^{-1}\rho_3$.

Now, we seek for a generalization of the three-step analysis above to the generic $N \geq 3$ case. Let $\rho_N = \mathrm{dist}\left(\mathbb{R}^n \backslash \mathrm{Capt}_N(\Xi), \Xi\right) > 0$. Being $\Xi \subseteq (\mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\delta_N))$, it follows that

$$\forall x \in \mathcal{C}_1(\Xi),\ \exists\, u_x \in U \colon \hat{f}(x, u_x) \in (\mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\rho_N)), \tag{32}$$

which, in view of Lemma 2.2, implies

$$\forall x \in \mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right),\ \exists\, u_x \in U \colon \hat{f}(x, u_x) \in (\mathcal{C}_N(\Xi) \backsim \mathcal{B}^n(\rho_N - \eta_0)), \tag{33}$$

for all $\eta_0 \in [0, \rho_N]$. Now, let us consider the sets $\mathcal{C}_i(\Xi)$ and $\mathcal{C}_{i+1}(\Xi)$, with $i \in \{1, \ldots, N-2\}$. By exploiting Lemma 2.2, we have that

$$\forall x \in \mathcal{C}_{i+1}(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right),\ \exists\, u_x \in U \colon \hat{f}(x, u_x) \in \mathcal{C}_i(\Xi) \oplus \mathcal{B}(\eta_i), \tag{34}$$

for all $\eta_i \in \mathbb{R}_{\geq 0}$. By using, again, Lemma 2.2, we obtain

$$\forall x \in \mathcal{C}_N(\Xi),\ \exists\, u_x \in U \colon \hat{f}(x, u_x) \in \mathcal{C}_{N-1}(\Xi), \tag{35}$$

From (33), (34), (35), thanks to Property 2.1, it follows that

$$\begin{aligned} &\mathcal{C}_N(\Xi) \cup \left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_{i+1}(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right)\right]\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right) \\ &\quad \subseteq \mathcal{C}_1\left(\mathcal{C}_{N-1}(\Xi) \cup \left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_i(\Xi) \oplus \mathcal{B}(\eta_i)\right]\right) \cup \left(\mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\rho_N - \eta_0)\right)\right). \end{aligned} \tag{36}$$

Moreover, if $\eta_0 \in (0, \rho_N)$, $\eta_1 < L_{\hat{f}_x}{}^{-1}\eta_0$ and $\eta_i < L_{\hat{f}_x}{}^{-1}\eta_{i-1}$, $\forall i \in \{1, \ldots, N-2\}$, with $\eta_i \in \mathbb{R}_{>0}$, $\forall i \in \{1, \ldots, N-2\}$, then the following inclusion holds

$$\begin{aligned} &\mathcal{C}_{N-1}(\Xi) \cup \left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_i(\Xi) \oplus \mathcal{B}(\eta_i)\right]\right) \cup \left(\mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\rho_N - \eta_0)\right) \\ &\quad \subset \mathcal{C}_N(\Xi) \cup \left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_{i+1}(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right)\right]\right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right). \end{aligned} \tag{37}$$

Finally, from the topology of the sets involved it turns out that

$$\begin{aligned}\mathrm{dist}\Bigg(&\mathbb{R}^n\backslash\left[\mathcal{C}_N(\Xi)\cup\left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_{i+1}(\Xi)\oplus\mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right)\right]\right)\cup\left(\mathcal{C}_1(\Xi)\oplus\mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right)\right],\\ &\left[\mathcal{C}_{N-1}(\Xi)\cup\left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_i(\Xi)\oplus\mathcal{B}(\eta_i)\right]\right)\cup\left(\mathrm{Capt}_N(\Xi)\backsim\mathcal{B}^n(\rho_N-\eta_0)\right)\right]\Bigg)\\ &\geq\min\left\{\rho_N-\eta_0,L_{\hat{f}_x}^{-1}\eta_0-\eta_1,\ldots,L_{\hat{f}_x}^{-1}\eta_{i-1}-\eta_i,\ldots,L_{\hat{f}_x}^{-1}\eta_{N-3}-\eta_{N-2},L_{\hat{f}_x}^{-1}\eta_{N-2}\right\}\end{aligned} \tag{38}$$

Choosing η_0 and $\eta_i,\ i\in\{1,\ldots,N-2\}$ such that all the N terms in the right-hand side of (38) are equal, then inequality (38) holds with positive right-hand side.

Hence, in view of Lemma 2.1, (36), (37) and (38) together imply that the set (extended N-steps capture basin)

$$\overline{\mathrm{Capt}}_N(\Xi,\overline{d}_N)\triangleq\mathcal{C}_N(\Xi)\cup\left(\bigcup_{i=1}^{N-2}\left[\mathcal{C}_{i+1}(\Xi)\oplus\mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_i\right)\right]\right)\cup\left(\mathcal{C}_1(\Xi)\oplus\mathcal{B}\left(L_{\hat{f}_x}^{-1}\eta_0\right)\right) \tag{39}$$

is $\overline{d}_N$-RC, with

$$\overline{d}_N=\left(\sum_{i=0}^{N-1}L_{\hat{f}_x}^{i}\right)^{-1}\rho_N\ =\frac{L_{\hat{f}_x}-1}{L_{\hat{f}_x}^{N}-1}\rho_N \tag{40}$$

■

Remark 2.1: Note that, if the target set Ξ is not ρ-RC, then the region $(\,\mathrm{Capt}_N(\Xi)\backslash\mathcal{C}_N(\Xi)\,)$ may even be not empty. In this case, for any initial condition in $(\,\mathrm{Capt}_N(\Xi)\backslash\mathcal{C}_N(\Xi)\,)$, the state cannot be driven to Ξ in exact N steps, but Ξ can be reached for some i with $i<N$ from the capture basin. Therefore, condition (14) (reachability in at most N steps) is less restrictive than requiring the exact N-steps controllability of Ξ. □

III. Problem Statement

The minimum-time problem for discrete-time system not affected by uncertainties has the following well known formulation: given an initial state $x_0\in\mathbb{R}^n$ and a target set $\Xi\subset\mathbb{R}^n$, find a sequence of control actions $\mathbf{u}\in\mathcal{M}_U$ which minimizes the time $T_{MT}(x_0|\Xi)$ such that $\hat{x}(x_0,T_{MT},\mathbf{u})\in\Xi$. In the following we will denote as $T_{MT}^o(x_0|\Xi)$ the minimum reach time.

The above formulation is commonly referred to as the open-loop approach to the minimum-time problem, that is, an optimal control sequence is determined on the basis of the particular initial state, relying on a nominal model of the controlled system. In the linear framework, it is well-known that the minimum-time problem admits a feedback solution, that is, it is possible to determine a control function $u = \kappa_{MT}(x|\Xi)$ such that $T_{MT}(\cdot|\Xi)$ is minimized for any possible initial state. We point out the minimum-time control law $\kappa_{MT}(\cdot|\Xi)$ is not, in general, unique. Therefore, for the sake of the present discussion, the notation $\kappa_{MT}(x|\Xi)$ will denote an arbitrary selection among the possible minimum-time feedback laws.

In nominal conditions, open-loop and feedback formulations are equivalent in the sense that a feedback solution is optimal if and only if for any initial states x_0 the control sequence $\mathbf{u}$ produced by the control $u = \kappa_{MT}(x, \Xi)$ along the systems'trajectory is optimal in the open-loop sense. On the other side, the feedback approach allows also to embed in the design of the controller some a priori information on the disturbances/uncertainties, yielding to minimum-time control laws with enhanced robustness properties. However, for a generic nonlinear system, it is very difficult to obtain an explicit minimum-time control function, even in the nominal case. Moreover, in practice, the search for a minimum-time open-loop sequence is performed over a compact set of sequences of finite length, subsuming a specified upper bound $N \in \mathbb{Z}_{>0}$.

A viable solution to alleviate the lack of robustness of open-loop approaches consists in solving, along system trajectories, a finite-time optimization problem in a receding horizon (RH) fashion. In the sequel, we will determine sufficient conditions (related, in particular, the controllability and reachability properties of the target set) under which the RH implementation guarantees the recursive feasibility of the optimization (that is, the robust postive invariance or the quasi-invariance of the feasible region) and the boundedness of the closed-loop trajectories.

Problem 3.1 (RH Nominal Minimum-Time Control): Given a compact admissible set $U \subset \mathbb{R}^m$ for the input of the system (1), a compact target set $\Xi \subset \mathbb{R}^n$, a finite integer $N \in \mathbb{Z}_{>0}$ and the nominal state-transition map $\hat{f}(x,u)$ of the system, at each time $t \in \mathbb{Z}_{\geq 0}$ determine a sequence $\mathbf{u}^o_{[t,t+N-1]} = \{u^o_t, u^o_{t+1}, \ldots, u^o_{t+N-1}\}$, in correspondence of the current state measurement x_t, such that:

$$T_{NMT}(x_t, \mathbf{u}^o_{[t,t+N-1]}|\Xi, N) = T^o_{MT}(x_t|\Xi),$$

with

$$T_{NMT}(x_t, \mathbf{u}_{[t,t+N-1]}|\Xi, N) \triangleq \min\Big\{\tau \in \{1,\ldots,N\} : \hat{x}(\tau, x_t, \mathbf{u}_{[t,t+\tau-1]}) \in \Xi\Big\}, \tag{41}$$

and

$$T^o_{MT}(x_t|\Xi) \triangleq \min_{\mathbf{u}_{[t,t+N-1]}\in U^N} \Big\{T_{NMT}(x_t, \mathbf{u}_{[t,t+N-1]}|\Xi, N)\Big\},$$

then apply to the plant the first element of $\mathbf{u}^o_{[t,t+N-1]}$ by setting $u_t = u^o_t$. □

If the Problem 3.1 is feasible in x_t, then u^o_t is a selection among the admissible minimum-time control actions for the current state, i.e., $u^o_t \in K_{MT}(x_t|\Xi)$.

This problem can be solved, in the discrete-time framework, by checking the feasibility of the target set constraint in (41). The feasibility check approach consists in embedding Problem 3.1 into a family of input-constrained minimum-distance problems as follows:

$$J^o_{MD}(x_t|\Xi, \tau) = \min_{\mathbf{u}_{[t,t+\tau-1]}\in U^\tau} \Phi\Big(\hat{x}(\tau, x_t, \mathbf{u}_{[t,t+\tau-1]}), \Xi\Big). \tag{42}$$

parametrized by the integer $\tau \in \mathbb{Z}_{>0}$. For a given τ, the minimizer is a fixed-length sequence $\mathbf{u}_{t,t+\tau-1}$ belonging to the compact set U^τ.

The feasible region for the original problem 3.1 is the basin of capture $\mathrm{Capt}_N(\Xi)$, which verifies

$$\mathrm{Capt}_N(\Xi) = \{x_t \in \mathbb{R}^n | \quad \exists\, \overline{\tau} \in \{1,\ldots,N\} : \, J^o_{MD}(x_t, \Xi, \overline{\tau}) \leq 0\}\,.$$

Then, assuming that $x_t \in \mathrm{Capt}_N(\Xi)$, at each time $t \geq 0$ the optimal time $T^o_{MT}(x_t|\Xi)$ is determined as the minimum among $\tau \in \{1,\ldots,N\}$ for which problem (42) yields to

$$J^o_{MD}(x_t|\Xi, \tau) \leq 0, \tag{43}$$

that is

$$T^o_{MT}(x_t|\Xi) = \min\{\tau : J^o_{MD}(x_t|\Xi,\tau) \leq 0\}$$

The condition $J^o_{MD}(x_t|\Xi,\tau) \leq 0$ represents a feasibility test aimed to ensure that x_t belongs to the τ-steps capture basin of Ξ.

Once the minimum-time $T^o_{MT}(x_t|\Xi)$ has been determined, we can take as a solution any control sequence which may steer the state to Ξ in $T^o_{MT}(x_t|\Xi)$ steps. A simple choice is

$$\mathbf{u}^o_{[t,t+T^o_{MT}(x_t|\Xi)-1]} = \arg \min_{\mathbf{u}_{[t,t+T^o_{MT}(x_t|\Xi)-1]} \in U^{T^o_{MT}(x_t|\Xi)}} J^o_{MD}\left(x_t \middle| \Xi, T^o_{MT}(x_t|\Xi)\right) \tag{44}$$

It is important to determine those conditions under which, starting from $x_0 \in \mathrm{Capt}_N(\Xi)$, the trajectories remain in the feasible set, in order to guarantee the solvability of the optimization at each time $t > 0$.

IV. RECURSIVE FEASIBILITY UNDER THE NOMINAL NONLINEAR MINIMUM-TIME CONTROL WITH ROBUST POSITIVELY CONTROLLABLE TARGET SET

In case the target set Ξ is robust positively controllable (i.e., there exists an admissible control law which renders Ξ RPI), then the $N-$steps basin of capture $\mathrm{Capt}_N(\Xi)$ coincides with the N-steps controllability set of Ξ. Moreover $\mathrm{Capt}_N(\Xi)$ is RPI under the minimum-time control, as formally stated by the following theorem.

Theorem 4.1 (Ξ ρ-RC $\rightarrow$ $\mathrm{Capt}_N(\Xi)$ RPI): Given a $\rho-$RC compact target set $\Xi \subset \mathbb{R}^n$, then the Nominal nonlinear Minimum-Time Control $\kappa_{MT}(x_t)$ guarantees the boundedness of the closed-loop trajectories within the set $\mathrm{Capt}_N(\Xi)$, for any initial condition $x_0 \in \mathrm{Capt}_N(\Xi)$ and for any admissible uncertainty realization $\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$, with $d_N = L^{-N}_{\hat{f}_x}\rho_N$.

Moreover, the closed-loop trajectories starting at time $t = 0$ from any point $x(0) = x_0 \in \mathrm{Capt}_N(\Xi)$ are ultimately bounded in the compact set

$$\Upsilon_N(\Xi, d_N) \triangleq \Xi \oplus \mathcal{B}\left(\frac{L^N_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} d_N\right) \subseteq \mathrm{Capt}_N(\Xi), \tag{45}$$

which is reached in finite time, for any possible realization of the uncertainties ($\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$), that is, $x(t, x_0, \mathbf{u}_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) \in \Upsilon_N(\Xi, ||\mathbf{d}_{[0,t-1]}||) \subseteq \Upsilon_N(\Xi, d_N)$, $\forall x_0 \in \mathrm{Capt}_N(\Xi)$, $\forall t \geq N$, $\forall \mathbf{d}_{[0,t-1]} : ||\mathbf{d}_{[0,t-1]}|| \leq d_N$. □

Proof: First, the boundedness of the trajectories can be proven by showing that $\mathrm{Capt}_N(\Xi)$ is RPI under the minimum-time control for any $\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$.

Indeed, applying recursively Theorem 2.3, it holds that

$$\mathcal{C}_i(\Xi) \supset \mathcal{C}_{i-1}(\Xi), \ \ \forall\, i > 1,$$

which yields to

$$\mathcal{C}_N(\Xi) \supset \mathcal{C}_j(\Xi), \ \ \forall\, j \in \{1, \ldots, N-1\},$$

that finally implies

$$\mathcal{C}_N(\Xi) \equiv \mathrm{Capt}_N(\Xi). \tag{46}$$

Moreover, thanks to Point $i)$ of Lemma 2.4, it holds that $\mathcal{C}_{N-1}(\Xi) \in \mathcal{C}_N(\Xi) \backsim \mathcal{B}^n(L_{\hat{f}_x}^{-N}\rho)$, which, thanks to (46), implies that $\forall x \in \mathrm{Capt}_N(\Xi)$, $u = \kappa_{MT}(x) : \ \hat{f}(x, u) + d \in \mathrm{Capt}_N(\Xi)$, $\forall d \in \mathcal{B}^n(L_{\hat{f}_x}^{-N}\rho)$ (that is, $\mathrm{Capt}_N(\Xi)$ is RPI under $\kappa_{MT}(x)$).

Now, we will prove the ultimate boundedness of the trajectories in the set $\Upsilon_N(\Xi, d_N)$. Let us consider the optimal minimum-time sequence $\mathbf{u}^o_{[0,N-1]}$, computed in open-loop at time $t = 0$ for the initial condition $x_0 \in \mathrm{Capt}(\Xi)$, and the correspondent nominal finite-time trajectory $\hat{x}(i, x_0, \mathbf{u}^o_{[0,i-1]}), \forall i \in \{1, \ldots, N\}$. The true evolution of the perturbed system, driven by $\mathbf{u}^o_{[0,N-1]}$, can be bounded by

$$|\hat{x}(i, x_0, \mathbf{u}^o_{[0,i-1]}) - x(t, x_0, \mathbf{u}^o_{[0,i-1]}, \mathbf{d}_{[0,i-1]})| \leq \frac{L_{\hat{f}_x}^i - 1}{L_{\hat{f}_x} - 1} ||\mathbf{d}_{[0,i-1]}||, \forall i \in \{1, \ldots, N\} \tag{47}$$

for any admissible realization of the uncertainties $\mathbf{d}_{[0,i-1]}$. In particular, since $\hat{x}(N, x_0, \mathbf{u}^o_{[0,N-1]}) \in \Xi$, we have that

$$x(N, x_0, \mathbf{u}^o_{[0,N-1]}, \mathbf{d}_{[0,N-1]}) \in \Xi \oplus \mathcal{B}^n \left(\frac{L_{\hat{f}_x}^N - 1}{L_{\hat{f}_x} - 1} ||\mathbf{d}_{[0,N-1]}|| \right) \subseteq \Upsilon_N(\Xi, d_N). \tag{48}$$

If, in addition, $||\mathbf{d}|| \leq L_{\hat{f}_x}^{-N}\rho$, then the closed-loop trajectories remain in the feasible set (in particular, $x_1 \in \mathrm{Capt}(\Xi)$). At time $t = 1$, the minimum-time control problem in x_1 can be solved with respect to a new open-loop sequence of controls $\mathbf{u}^o_{[1,N]}$. Proceeding along the same lines as above, it is possible to bound the finite-time perturbed evolution of the system as follows

$$x(N+1, x_1, \mathbf{u}^o_{[1,N]}, \mathbf{d}_{[1,N]}) \in \Xi \oplus \mathcal{B}^n\left(\frac{L_{\hat{f}_x}^N - 1}{L_{\hat{f}_x} - 1}||\mathbf{d}_{[1,N]}||\right) \subseteq \Upsilon_N(\Xi, d_N). \tag{49}$$

Moreover, $x_2 \in \mathrm{Capt}(\Xi)$ for any possible uncertainty realization. By induction, it follows that the closed loop trajectories remain bounded in $\Upsilon_N(\Xi, d_N)$ for any $t \geq N$. ■

Now, our analysis will be extended to the case in which Ξ is not one step robust positively controllable. In this regard, the result of Theorem 2.1, obtained by set-invariance theoretic analysis, implies the existence of a (possibly non unique) control law which renders the set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ a $\overline{d}_N$-RPI set, with $\overline{d}_N$ defined in (16). However, for non null-uncertainties, the set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ is such that $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N) \supset \mathrm{Capt}_N(\Xi)$. Recalling the the feasible region for Problem 3.1 is as small as $\mathrm{Capt}_N(\Xi)$, in the set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N) \backslash \mathrm{Capt}_N(\Xi)$ the finite-time RH problem does not admit a solution (i.e., the feasibility check (43) fails). Therefore, we seek for a backup control law to be applied in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ when the minimum-time problem is not solvable, but capable to keep the trajectories bounded in the extended N-steps capture basin $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$. Notably, by Theorem 2.1 we have established the existence of a control law, compliant with the input constraints, capable to achieve this task. We are now on the way to show how such a robust control law can be obtained.

V. ROBUST NONLINEAR MINIMUM-TIME CONTROL LAWS WITH NON ROBUST POSITIVELY CONTROLLABLE TARGET SETS

In the following, we are going to describe a modified minimum-time control scheme, which will be referred to as Robustified Nonlinear Minimum-Time Control (RNMT), that guarantees the quasi-invariance of the feasible region, despite bounded uncertainties and with mild assumptions on the target set Ξ. The RNMT control, computed online according to Procedure 5.1 below, consists in a control schemes that switches between the regular minimum-time control and a backup control action when transitory unfeasibility occurs; hence, in nominal conditions, the RNMT corresponds to the convential receding horizon minimum-time control, being the feasible region invariant in this case. Conversley, in perturbed conditions, the backup control action is taken from a buffer in which a time-optimal control sequence had been saved after the most recent feasible optimization. The key point of this procedure is that feasbility is recovered before buffer overrun occurs. As long as the system's state enters the feasible region, an optimal solution is computed and the buffer is reinitialized with a new sequence, that will be used to cope with future infeasibility occurences.

Next, the RNMT scheme is formalized by a listed procedure describing the actions to be performed by the controller.

Procedure 5.1 (RNMT): Let the controller be equipped with two buffers: $i)$ $\mathbf{u}^b \in \mathbb{R}^m \times N$, used to store a sequence of N control actions; $ii)$ $T^b \in \mathbb{Z}$, that stores the time instant in which the sequence stored in $\mathbf{u}^b$ had been computed. Moreover, let us denote as $\leftarrow$ a data assignment operation. Given the buffer (memory array) $\mathbf{u}^b$, let $u^b(i)$ represent the i-th element of the array, with $i \in \{1, \dots, N\}$.

Initialization

1 Assuming that, at time instant $t = 0$, the initial condition verifies $x_0 \in \mathrm{Capt}_N(\Xi)$, solve the nominal minimum-time Problem 3.1 obtaining an optimal control sequence $\mathbf{u}^o_{0,N-1}$;

2 store $\mathbf{u}^b \leftarrow \mathbf{u}^o_{0,N-1}$;

3 store $T^b \leftarrow 0$;

4 apply $u_o = u^b(1)$ to the plant.

On-line Control Computation

1 for $t \in \mathbb{Z}_{>0}$:

given x_t, perform the feasibility test (43) for $\tau \in \{1, \ldots, N\}$;

if exists at least one τ for which $J^o_{MD}(x_t|\Xi, \tau) \leq 0$, then :

compute $\mathbf{u}^o_{t,t+N-1}$ with (44) ;

overwrite the buffer $\mathbf{u}^b \leftarrow \mathbf{u}^o_{t,t+N-1}$;

set $T^b \leftarrow t$;

end if;

apply $u_t = u^b(t - T^b + 1)$ to the plant;

end for;

□

The following theorem formally states the recursive feasibility property (that is, the quasi-invariance of the feasible region) of the RNMT scheme for bounded additive uncertainties.

Theorem 5.1 (Quasi-invariance of the feasible set): Given a compact target set $\Xi \subset \mathbb{R}^n$ (possibly not robustly controllable) such that $\Xi \subseteq \mathrm{Capt}_N(\Xi) \backsim \mathcal{B}^n(\rho_N)$, then, for any initial condition $x_0 \in \mathrm{Capt}_N(\Xi)$, the RNMT control $u_t = \kappa_{RNMT}(t, x_t)$ guarantees that the closed-loop system's trajectory is ultimately contained in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ for any admissible uncertainty realization $\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(d_N)}$, with $\overline{d}_N$ given by (16). Moreover, the compact set $\Upsilon_N(\Xi, \overline{d}_N) \subseteq \mathrm{Capt}_N(\Xi)$ (defined in (51) below) is reached in finite-time from x_0 and is quasi-RPI in closed-loop, for any possible realization of the uncertainties. □

Proof: The quasi-invariance result can be obtained by induction: given $x_0 \in \mathrm{Capt}_N(\Xi)$, the optimal minimum-time sequence $\mathbf{u}^o_{[0,T^o_{MT}(x_0)-1]}$ computed at time $t = 0$ yields to $\hat{x}(T^o_{MT}(x_0), x_0, \mathbf{u}^o_{[0,T^o_{MT}(x_0)-1]}) \in \Xi$ for the nominal trajectory. Now, the true trajectory departing from x_0 is bounded in a closed envelope around the nominal one, for all $t \in \{1, \dots, T^o_{MT}(x_0)\}$

$$\begin{aligned}
&\left| x(t, x_0, \mathbf{u}^o_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) - \hat{x}(t, x_0, \mathbf{u}^o_{[0,t-1]}) \right| \\
&\quad \le \frac{L^t_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} \left(\eta_u(\overline{u}) + ||\mathbf{d}_{[0,t-1]}|| \right), \\
&\quad \le \frac{L^t_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} \overline{d}_N,
\end{aligned} \tag{50}$$

for any bounded sequence $\mathbf{d}_{[0,T^o_{MT}(x_0)-1]} \in \left(\mathcal{B}^n(\overline{d}_N)\right)^{T^o_{MT}(x_0)}$. At time $t = T^o_{MT}(x_0)$ we have

$$\begin{aligned}
&\left| x(T^o_{MT}(x_0), x_0, \mathbf{u}^o_{[0,T^o_{MT}(x_0)-1]}, \mathbf{d}_{[0,T^o_{MT}(x_0)-1]}) - \hat{x}(T^o_{MT}(x_0), x_0, \mathbf{u}^o_{[0,T^o_{MT}(x_0)-1]}) \right| \\
&\quad \le \frac{L^{T^o_{MT}(x_0)}_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} \overline{d}_N.
\end{aligned}$$

Now, defining the set

$$\Upsilon_N(\Xi, \overline{d}_N) \triangleq \Xi \oplus \mathcal{B}^n \left(\max_{j \in \{1,\dots,N\}} \left\{ \frac{L^j_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1} \overline{d}_N \right\} \right) \tag{51}$$

it holds that

$$x_{T^o_0} = x\left(T^o_{MT}(x_0), x_0, \mathbf{u}^o_{[0,T^o_{MT}(x_0)-1]}\right) \in \Upsilon_N(\Xi, \overline{d}_N) \subseteq \mathrm{Capt}_N(\Xi) \tag{52}$$

under the closed-loop RNMT control. In view of the inclusion (52) , at time $T^o_0 = T^o_{MT}(x_0)$ a feasible time-optimal sequence can be computed, and the same arguments can be used to show that the perturbed closed-loop trajectory departing from $x_{T^o_0}$ verifies $x\left(T^o_{MT}(x_{T^o_0}), x_{T^o_0}, \mathbf{u}_{[T^o_0, T^o_{MT}(x_{T^o_0}))-1]}\right) \in \Upsilon_N(\Xi, \overline{d}_N)$.

Therefore, the return-time to $\mathrm{Capt}_N(\Xi)$ from a point x_0 with minimum-time $T^o_0 = T^o_{MT}(x_0)$ never exceedes the minimum-time itself. This key result establishes the inherent buffer-underrun consistency of the RNMT scheme. Proceeding by induction, it follows that the compact set $\Upsilon_N(\Xi, \overline{d}_N) \subseteq \mathrm{Capt}_N(\Xi)$ and the feasible set $\mathrm{Capt}_N(\Xi)$ are both quasi-RPI (see Definition 2.8) under the closed-loop trajectories with maximum return-time N.

Now, we will prove that the trajectories are confined in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ during the transitory departures from $\mathrm{Capt}_N(\Xi)$. The set $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$, defined in (15), can be equivalently expressed as

$$\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N) = \mathcal{C}_N(\Xi) \cup \left(\bigcup_{i=1}^{N-2} \left[\mathcal{C}_{i+1}(\Xi) \oplus \mathcal{B}\left(\epsilon_{i+1}\right) \right] \right) \cup \left(\mathcal{C}_1(\Xi) \oplus \mathcal{B}\left(\epsilon_1\right) \right) \tag{53}$$

where

$$\epsilon_i \triangleq L_{\hat{f}_x}^{-1} \eta_{i-1} = \frac{L_{\hat{f}_x}^{N-i} - 1}{L_{\hat{f}_x} - 1} \overline{d}_N,$$

for all $i \in \{1, \ldots, N-1\}$. Since by (50) it holds that

$$\left| x(i, x_0, \mathbf{u}^o_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) - \hat{x}(t, x_0, \mathbf{u}^o_{[0,t-1]}) \right| \leq \epsilon_{N-i}, \tag{54}$$

for all $i \in \{1, \ldots, N-1\}$, and and $\hat{x}(i, x_0, \mathbf{u}^o_{[0,t-1]} \in \mathcal{C}_{N-i})$, $\forall i \in \{1, \ldots, N-1\}$ then we can conlude that

$$x(i, x_0, \mathbf{u}^o_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) \in \mathcal{C}_{N-i}(\Xi) \oplus \mathcal{B}\left(\epsilon_{N-i}\right) \subseteq \overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N), \ \ \forall i \in \{1, \ldots, N-1\}. \tag{55}$$

Proceeding by induction, it follows that the closed-loop trajectories are ultimately confined in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$. ■

Notice that the ultimate confinement property in $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$, together with the quasi-robust positive invariance of the compact set $\Upsilon_N(\Xi, \overline{d}_N)$, they by themselves do not imply the ultimate boundedness of the trajectories, since $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ can be unbounded. The boundedness in a compact subset of $\overline{\mathrm{Capt}}_N(\Xi, \overline{d}_N)$ can be proven by invoking the further Assumption 2 and by exploiting the the presence of input contraints.

Corollary 5.1 (Ultimate boundedness): If the nominal transition function of the system, $\hat{f}$, verifies, in addition to Assumptions of Theorem 5.1, the further Assumption 2, ($\mathbf{d} \in \mathcal{M}_{\mathcal{B}^n(\overline{d}_N)}$), then the closed-loop trajectories under the RNMT control are ultimately bounded in a compact set $\Lambda_N(\Xi, \overline{d}_N, \overline{u})$ (defined in (56) below) for any initial condition $x_0 \in \mathrm{Capt}_N(\Xi)$, that is: $x(t, x_0, \mathbf{u}_{[0,t-1]}, \mathbf{d}_{[0,t-1]}) \in \Lambda_N(\Xi, \overline{d}_N, \overline{u})$, $\forall x_0 \in \mathrm{Capt}_N(\Xi, \overline{d}_N)$, $\forall t \geq N$, $\forall \mathbf{d}_{[0,t-1]}$: $||\mathbf{d}_{[0,t-1]}|| \leq \overline{d}_N$. □

Proof: First, note that the set $\Upsilon_N(\Xi, \overline{d}_N)$ is reached in finite-time (in at most N steps). Hence, for some $k \in \{1, \ldots, N\}$ we have that $x_k \in \Upsilon_N(\Xi, \overline{d}_N)$. Moreover, by the quasi-invariance property, the trajectories departing from x_k are guaranteed to enter again in $\Upsilon_N(\Xi, \overline{d}_N)$ in at most further N steps. Let us analyze the closed-loop perturbed trajectory x_{k+j} for the worst-case-length interval in which the trajectories may live outside $\Upsilon_N(\Xi, \overline{d}_N)$. We have that, for any $j \in \{1, \ldots, N\}$,

$$\begin{aligned} |x(k+j, x_k, \mathbf{u}_{[k,k+j-1]}, \mathbf{d}_{[k,k+j-1]})| &\leq L^j_{\hat{f}_x}|x_k| + \frac{L^j_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1}\left(\eta_u(||\mathbf{u}_{[k,k+j-1]}||) + ||\mathbf{d}_{[k,k+j-1]}||\right) \\ &\leq L^j_{\hat{f}_x}|x_k| + \frac{L^j_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1}\left(\eta_u(\overline{u}) + \overline{d}\right). \end{aligned}$$

where $\overline{u} \triangleq \max_{u \in U}\{|u|\}$ Therefore, for any $j \in \{1, \ldots, N\}$, the trajectories remain bounded in

$$\Lambda_N(\Xi, \overline{d}_N, \overline{u}) \triangleq \overline{\mathrm{Capt}}_N(\Xi) \cap \Upsilon_N(\Xi, \overline{d}_N, \overline{u}) \oplus \mathcal{B}^n\left(\max_{j \in \{1,\ldots,N\}}\left\{\frac{L^j_{\hat{f}_x} - 1}{L_{\hat{f}_x} - 1}\left(\eta_u(\overline{u}) + \overline{d}_N\right)\right\}\right). \quad (56)$$

The ultimate boundedness of the closed-loop trajectories follows from the compactness of $\Lambda_N(\Xi, \overline{d}_N, \overline{u})$. ■

VI. Academic Example

To show the effectiveness of the method, we will apply the robustified nonlinear minimum-time control to the following discrete-time open-loop unstable system:

$$\begin{cases} x_{(1)_{t+1}} &= x_{(1)_t}\left[1.1 + 0.4\,\mathrm{sign}(\mathrm{x}_{(1)_t})u_t\right] + (x_{(2)_t}^2 + 2)^{-1}u_t + d_{(1)_t} \\ x_{(2)_{t+1}} &= 0.94\,x_{(2)_t} - x_{(2)_t}u_t + d_{(2)_t} \end{cases}, t \in \mathbb{Z}_{\geq 0}. \quad (57)$$

subjected to the input constraint $|u_t| < 2$. The subscripts (i), $i \in \{1, 2\}$ in (57) denote the i-th component of $x_t \in \mathbb{R}^2$, while $d_t \in \mathbb{R}^2$ is a bounded exogenous disturbance. First, we will prove that the nominal transition function of the system is Lipschitz continuous with respect to the state variables.

Proposition 6.1: The nonlinear transition function $\hat{f}(x,u)\ :\ \mathbb{R}^2 \times [-R,R] \ \rightarrow\ \mathbb{R}^2$, with $\hat{f}(x,u) = \left(\hat{f}_{(1)}(x_{(1)},x_{(2)},u), \hat{f}_{(2)}(x_{(2)},u)\right)$ given by

$$\begin{aligned} \hat{f}_{(1)}(x_{(1)},u) &= x_{(1)}\left[1.1 + 0.4\,\mathrm{sign}(\mathrm{x}_{(1)})u\right] + ({x_{(2)}}^2+2)^{-1}u, \\ \hat{f}_{(2)}(x_{(2)},u) &= 0.94\,x_{(2)} -\ x_{(2)}\,u, \end{aligned} \tag{58}$$

is Lipschitz continuous in x, uniformly for $u \in [-2,2]$.

Proof: Consider the scalar function $\hat{f}_{(1)}(x_{(1)},x_{(2)},u) : \mathbb{R}^2 \times [-2,2] \rightarrow \mathbb{R}$. Then, given two points $(x_{(1)},x_{(2)})$ and $(x^{'}_{(1)},x^{'}_{(2)}) = (x_{(1)}+\delta x_{(1)}, x_{(2)}+\delta x_{(2)})$, for any fixed u we have

$$\begin{aligned} &\left|\hat{f}_{(1)}(x^{'}_{(1)},{x_{(2)}}^{'},u) - \hat{f}_{(1)}(x_{(1)},{x_{(2)}}_t,u)\right| \\ &\leq \Bigg|(x_{(1)}+\delta x_{(1)})\left[1.1+0.4\,\mathrm{sign}(x_{(1)}+\delta x_{(1)})u\right] + \frac{1}{(x_{(2)}+\delta x_{(2)})^2+2}u \\ &\quad -x_{(1)}\left(1.1+0.4\,\mathrm{sign}(x_{(1)})u\right) + \frac{1}{{x_{(1)}}^2+2}u\Bigg| \\ &\leq \Big|(x_{(1)}+\delta x_{(1)})\left(1.1+0.4\,\mathrm{sign}(x_{(1)}+\delta x_{(1)})u\right) - x_{(1)}\left(1.1+0.4\,\mathrm{sign}(x_{(1)}u)\right) \\ &\quad + \left((x_{(2)}+\delta x_{(2)})^2+2\right)^{-1}u - ({x_{(2)}}^2+2)^{-1}u\Big| \end{aligned}$$

Being $\left|\dfrac{\partial}{\partial x_{(2)}}\dfrac{1}{(x_{(2)})^2+2}\right| < 0.23$, $\forall x_{(2)} \in \mathbb{R}$, we have that

$$\begin{aligned} &\left|\hat{f}_{(1)}(x^{'}_{(1)},{x_{(2)}}^{'},u) - \hat{f}_{(1)}(x_{(1)},{x_{(2)}}_t,u)\right| \\ &\leq \left|x_{(1)}0.4\left(\mathrm{sign}(x_{(1)}+\delta x_{(1)}) - \mathrm{sign}(x_{(1)})\right)u + \delta x_{(1)}\left(1.1+0.4\,\mathrm{sign}(x_{(1)}+\delta x_{(1)})\right)u\right| \\ &\quad +0.23|\delta x_{(2)}u| \\ &\leq \left|x_{(1)}\right|\left|\mathrm{sign}(x_{(1)}+\delta x_{(1)}) - \mathrm{sign}(x_{(1)})\right|0.8 + \left|\delta x_{(1)}\right|3 + 0.46|\delta x_{(2)}| \end{aligned}$$

Note that, if $\left|x_{(1)}\right| > \left|\delta x_{(1)}\right|$, then the first addend in the right-hand side of the last inequality becomes null, and therefore $\left|\hat{f}_{(1)}(x^{'}_{(1)},x^{'}_{(2)},u) - \hat{f}_{(1)}(x_{(1)},x_{(2)},u)\right| \leq 3\left|\delta x_{(1)}\right| + 0.46|\delta x_{(2)}|$. Conversely, when $\left|x_{(1)}\right| \leq \left|\delta x_{(1)}\right|$, we have that $\left|\hat{f}_{(1)}(x^{'}_{(1)},x^{'}_{(2)},u) - \hat{f}_{(1)}(x_{(1)},x_{(2)},u)\right| \leq 4.6\left|\delta x_{(1)}\right| + 0.46|\delta x_{(2)}|$. Moreover, it holds that $\left|\hat{f}_{(2)}(x^{'}_{(2)},u) - \hat{f}_{(2)}(x_{(1)},u)\right| \leq 2.94\left|\delta x_{(2)}\right|$. Finally, the global upper bound for the Lipschitz constant of the transition function $\hat{f}$ is $L_{f_x} = 4.6$.

■

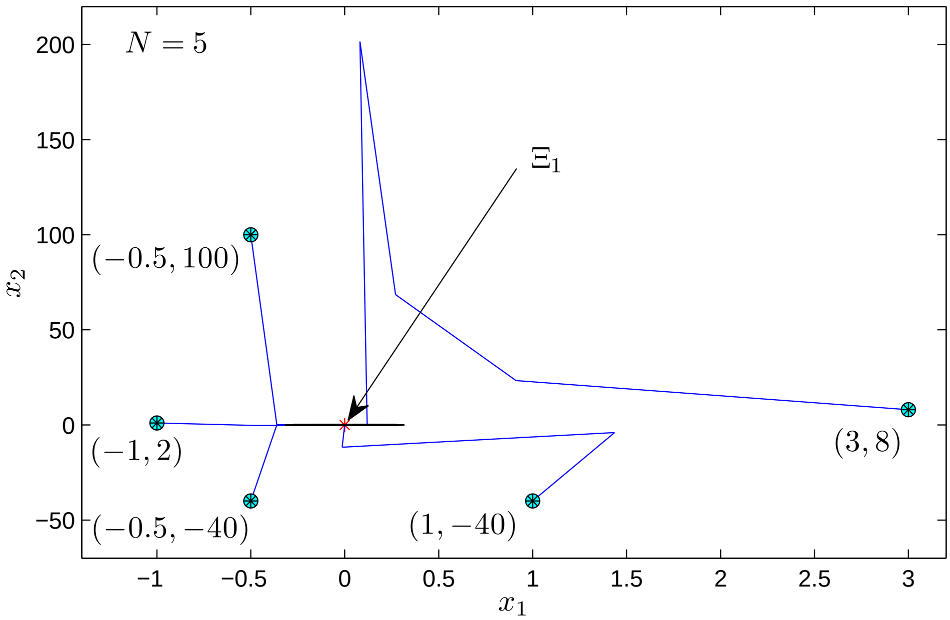

Fig. 2. Sample closed-loop trajectories under th RMNT control in nominal conditions

Figure 2 shows some sample closed-loop trajectories in nominal conditions (i.e., $d_t = 0, \forall t \in \mathbb{Z}_{\geq 0}$) for $N = 5$ and with target set

$$\Xi_1 = \left\{ x \in \mathbb{R}^2 : x^T \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} x \leq 1 \right\}.$$

The RNMT strategy, in the nominal case, steers the state into the target set in minimum-time.

Being Ξ_1 robustly controllable, then the inclusion $\Xi \subset \mathrm{Capt}_5(\Xi)$ holds; therefore, the theoretical results obtained so far can be used to assert the boundability of the trajectories by RNMT control in the uncertain/pertubed case. Since a direct explicit computation of the set $\mathrm{Capt}_5(\Xi)$ is difficult to obtain, even for the small-dimensional system of this example, in general it is very difficult to compute good approximation of the maximal admissible uncertainty, that has to be evaluated by simulations. In presence of bounded exogenous perturbations $||d_t|| \leq 10\sqrt{2}, \forall t \in \{0, \ldots, 100\}$, we will analyze the trajectory departing from the feasible initial condition $x_0 = (3, 8)$. Figure 3 shows that the RNMT control keeps the system's evolution bounded despite the disturbance. The values of the minimimum-time function returned by the

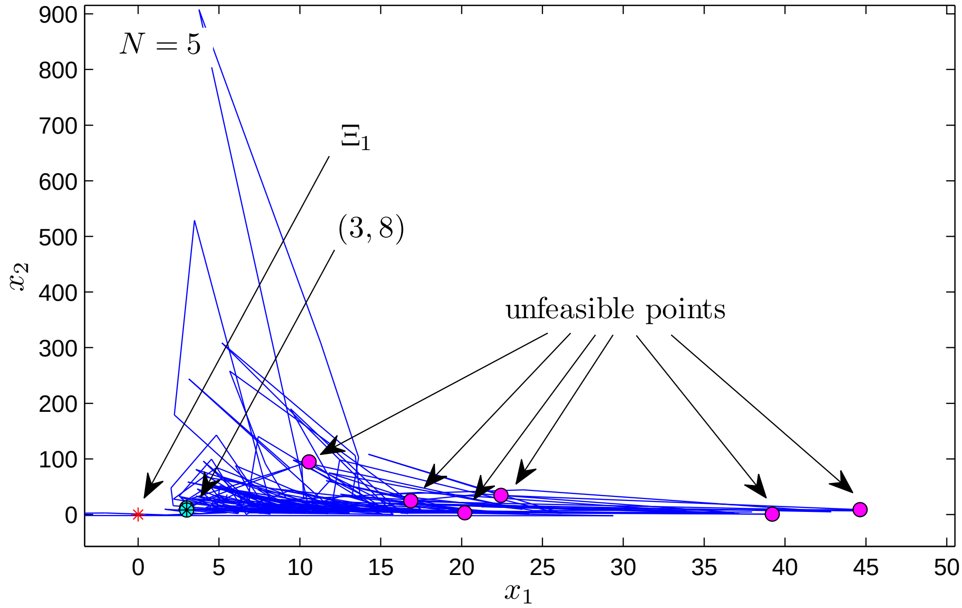

Fig. 3. Cloud of points obtained by simulating the closed-loop system with bounded perturbations. The transitory unfeasibilities occurring in presence of disturbances are marked by circles. The feasibility is always recovered in finite-time.

optimization performed along the system's trajectory are shown in Figure 4.

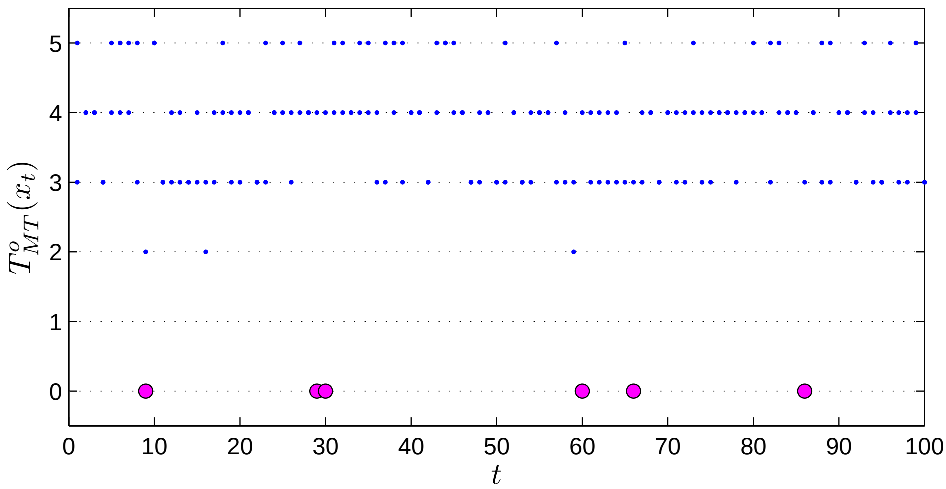

Fig. 4. Values of the minimum-time function retruned by the optimization along the system's trajectory. The circles represent unfeasible optimization, that is, the target set cannot be reached within N steps from the correspondent state.

Several unfeasible points have been reached during the experiment. We remark that the convenional minimum-time control would not have been able to cope with those unfeasibilities.

To complete the analysis, nominal trajectories obtained with a non robust positively controllable target Ξ_2, given by

$$\Xi_2 = \left\{ x \in \mathbb{R}^2 : \left(x - \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} \right)^T \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \left(x - \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} \right) \leq 1 \right\}.$$

are shown in Figure 5, for $N = 20$.

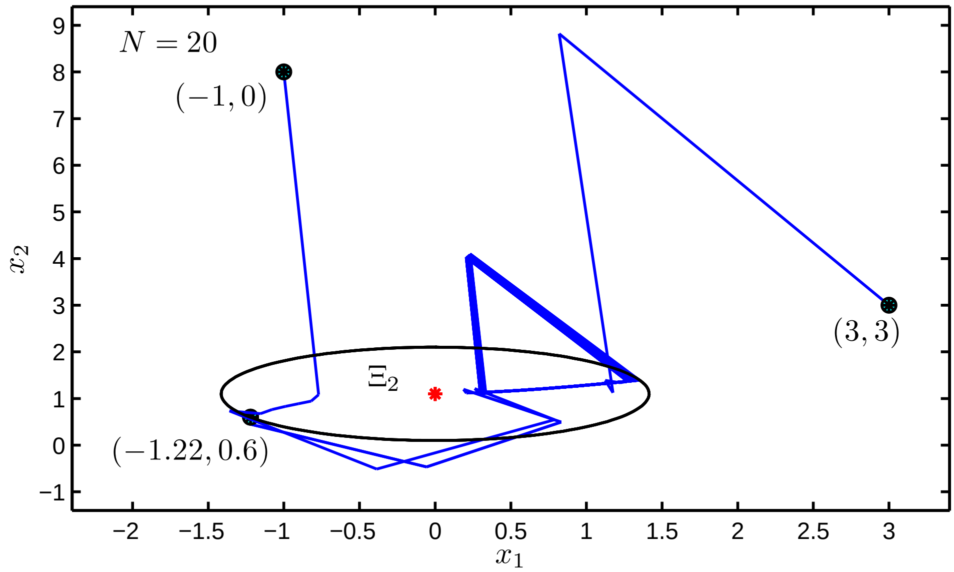

Fig. 5. Sample closed-loop trajectories under the RMNT control in nominal conditions, with a non robustly controllable target set Ξ_2. From the initial point $x_0 = (-1.22, 0.6) \in \Xi_2$ the state cannot be kept inside Ξ_2 with the available control input. The trajectories asymptotically reach a triangle-shaped limit-cycle condition which temporarily leaves the target set

The trajectories departing from all the considered initialization points asymptotically reach a triangle-shaped limit-cycle condition, that temporarily exists from the target set.

Even in this case, the RNMT can face exogenous perturbations, as shown in Figure 6 , where a bounded disturbance ($||d_t|| \leq \sqrt{2}, \forall t \in \{0, \ldots, 100\}$) has been simulated.

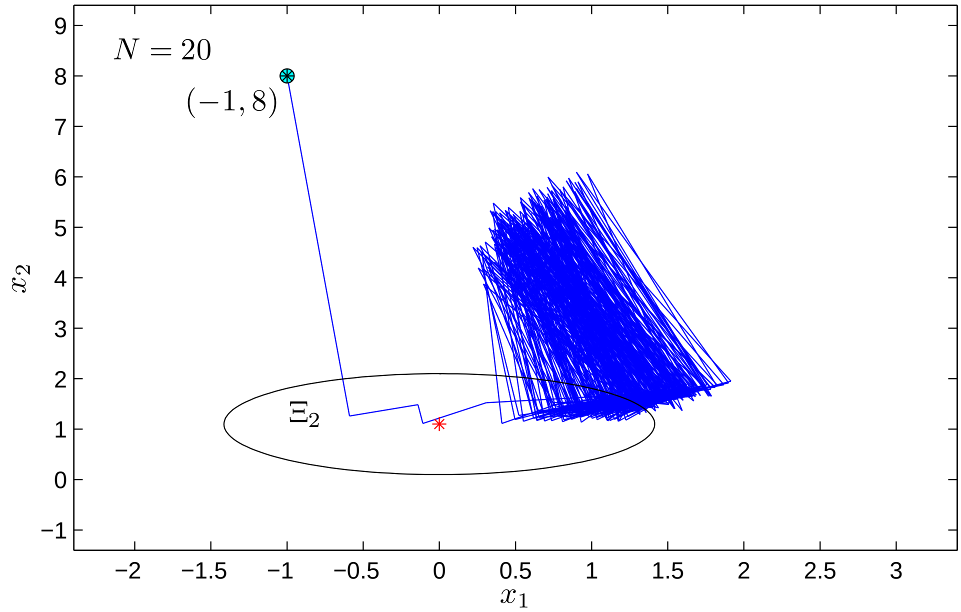

Fig. 6. Sample closed-loop trajectory under th RMNT control with bounded perturbations. The trajectory remains bounded in perturbed conditions despite a non robust positively controllable target set has been used

VII. Conclusion

In this work we have proposed a robustified minimum-time control scheme for nonlinear discrete-time systems with input constraints that admits non robustly controllable target sets. Given a Lipschitz nonlinear transition map with hard constraints on control inputs, the reachability properties of the target set have been used to assess the robustness of the minimum-time control. In particular, the recursive feasibility of the scheme is preserved with non robust positively controllable target set and in presence of bounded exogenous disturbances.

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